S-LIMITS AND @-SUMMABILITY

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ABSTRACT. Certain classes of subspaces of m are examined that are the domains of uniqueness of certain Banach type limits. These subspaces are identified as the bounded convergence domains of @-summability. As a corollary it is shown that the closure of bs in m is f_0 . Also we characterize those matrices whose convergence domains include these spaces in a manner analogous to Lorentz' characterization of strongly regular matrices.

S-limits. Throughout this paper, S will denote a fixed regular matrix with $\|S\| = 1$. We shall limit our discussion to sequences and matrices of real numbers. A S -limit is a linear functional, L , on m , the space of bounded sequences, that satisfies: (i) $L(x) \geq 0$ if $x_n \geq 0$, $n = 1, 2, \dots$; (ii) $L(\bar{1}) = \bar{1}$ ($\bar{1}$ is the constant sequence of 1's); (iii) $L(Sx) = L(x)$ for all $x \in m$. A Banach limit is a T -limit where T is the translation matrix, i.e. $(Tx)_n = x_{n+1}$. A matrix B is S -invariant if $B(S - I)$ maps every bounded sequence to a null sequence. (I is the identity matrix.)

S -limits need not exist for every regular matrix S . Let $S = (a_{nk})$ where $a_{nk} = 0$ if $k > 2n$ and $a_{nk} = 1$ if $k = 2n$. For $1 \leq k \leq 2n$, $a_{nk} = -1/n$ if k is odd, $a_{nk} = 1/n$ if k is even. S is regular, but for $x = \{1, 0, 1, 0, \dots\}$, we have $Sx = \{-1, -1, -1, \dots\}$. Now if L were an S -limit, we would have $0 \leq L(x) = L(Sx) = -1$. Thus S -limits need not exist.

THEOREM 1. *If there is a nonnegative regular S -invariant matrix, then S -limits exist.*

PROOF. Suppose B is such a matrix. Define a functional q on m by $q(x) = \text{Lim sup } Bx$. Then q is sublinear and nonnegatively homogeneous. For $x \in c$ we have, by the regularity of B , that $\text{Lim } x = q(x)$. By the Hahn-Banach Theorem, we can extend Lim to a linear functional, L , on m s.t. $L(x) \leq q(x)$ for all $x \in m$. Thus $-q(-x) \leq -L(-x) = L(x) \leq q(x)$, for all $x \in m$.

Since B is nonnegative, for $x_n \geq 0$, $n = 1, 2, \dots$, we have $0 \leq \text{Lim inf } Bx = -q(-x) \leq L(x)$. So L is nonnegative. Also for $x \in m$, $\text{Lim } B(S - I)x = 0$; thus $-q(-(S - I)x) = 0 = q((S - I)x)$ which proves $L((S - I)x) = 0$. So L is an S -limit.

We note that the Cesàro matrix is a nonnegative regular T -invariant matrix;

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hence, as a corollary, we have that Banach limits exist.

For a particular S , the determination of the existence of an S -invariant matrix is facilitated by Schur's Theorem (see, for example, [8, p. 20]) which gives necessary and sufficient conditions for $B(S - I)$ to map m to n , the space of null sequences.

In the case where S -limits exist for a particular regular matrix S , we shall consider the following sets which are the domains of uniqueness of these limits:

$$\Lambda(S) = \{x \in m: L(x) = L'(x) \text{ for every pair of } S\text{-limits } L, L'\},$$

$$\Theta(S) = \{x \in m: L(x) = 0 \text{ for every } S\text{-limit } L\}.$$

Note that $\Lambda(S)$ and $\Theta(S)$ are closed subspaces of m since each S -limit L is nonnegative, hence continuous.

We shall identify $\Theta(S)$ and $\Lambda(S)$, in certain cases, as the bounded convergence domains of the method of summability introduced in [3].

$\mathcal{Q}$ -summability. For $\nu = 1, 2, \dots$, let $A^\nu = (a_{nk}^\nu)$ be an infinite matrix of real numbers. Let $\mathcal{Q}$ denote the sequence of matrices $\{A^\nu\}$. A sequence $x = \{x_n\}$ is $\mathcal{Q}$ or $\{A^\nu\}$ -summable to a number l if $\text{Lim } A^\nu x = l$ uniformly in $\nu = 1, 2, \dots$. We say $\mathcal{Q}x$ converges to l and write $\text{Lim}_{\mathcal{Q}} x = l$. $\mathcal{Q}$ is said to be regular if $\mathcal{Q}x$ converges to l whenever x is convergent to l .

$$c_{\mathcal{Q}} = \{x: \mathcal{Q}x \text{ converges}\} \text{ and}$$

$$n_{\mathcal{Q}} = \{x: \mathcal{Q}x \text{ converges to } 0\}.$$

For a fixed regular matrix S , let Σ denote the sequence of matrices $\{S^0, S^1, S^2, \dots\}$, where S^ν denotes the product of S with itself ν times. We shall use the notation $A \cdot \Sigma$ to denote the sequence of matrices $\{R^p\}$ where $(R^p x)_n = \Sigma a_{nk}(S^k x)_p$. (Throughout, all sums run from $k = 1$ to ∞ unless otherwise noted.) Note that for T , the translation matrix, $(T^k x)_p = (T^p x)_k$, so if $S = T$, $A \cdot \Sigma$ reduces to the sequence of ordinary matrix products of A and powers of T .

THEOREM 2. *Suppose A is regular. If A is T -invariant, then $A \cdot \Sigma$ is S -invariant, that is $A \cdot \Sigma(S - I)x$ converges to 0 for all $x \in m$.*

PROOF. Let $\mathcal{R} = A \cdot \Sigma$. We show $\mathcal{R}$ is S -invariant. For $x \in m$,

$$(R^p(S - I)x)_n = \Sigma a_{nk}(S^k(S - I)x)_p = \Sigma a_{nk}[(S^{k+1}x) - (S^k x)_p].$$

So

$$\begin{aligned} |(R^p(S - I)x)_n| &\leq |\Sigma a_{n,k-1}(S^k x)_p - \Sigma a_{nk}(S^k x)_p| \\ &\leq \Sigma |a_{n,k-1} - a_{nk}| \|S^k\| \|x\|. \end{aligned}$$

Since A is T -invariant, $\Sigma |a_{n,k-1} - a_{nk}| \rightarrow 0$ as $n \rightarrow \infty$ by Schur's Theorem.

Hence $(R^p(S - I)x)_n \rightarrow 0$ as $n \rightarrow \infty$ uniformly in p . This shows $A \cdot \Sigma$ is S -invariant.

Thus we have the following information. If A is a nonnegative regular matrix which is T -invariant and S -invariant, then Theorem 1 states S -limits exist and Theorem 2 states $A \cdot \Sigma$ is S -invariant.

THEOREM 3. *If A is regular, then $n_{A \cdot \Sigma} \cap m \subset \Theta(S)$.*

PROOF. Suppose $x \in n_{A \cdot \Sigma} \cap m$. Let $t_n = \{t_n^\nu: \nu = 1, 2, \dots\}$ where $t_n^\nu = \sum a_{nk}(S^k x)_\nu$. Thus $t_n^\nu \rightarrow 0$ as $n \rightarrow \infty$ uniformly in $\nu = 1, 2, \dots$, that is $t_n \rightarrow 0$ in $(m, \|\cdot\|_\infty)$. Let L be any S -limit. L is continuous on m so $L(t_n) \rightarrow 0$ as $n \rightarrow \infty$.

We shall now show $L(t_n)$ tends to $L(x)$. Now, for n fixed we have

$$\left| t_n^\nu - \sum_{k=1}^N a_{nk}(S^k x)_\nu \right| \leq \|x\| \sum_{k=N+1}^{\infty} |a_{nk}| \|S\|^k \leq \|x\| \sum_{k=N+1}^{\infty} |a_{nk}|$$

since $\|S\| = 1$. For each n , $\sum_{k=N+1}^{\infty} |a_{nk}| \rightarrow 0$ as $N \rightarrow \infty$ since A is regular. Thus $\sum_{k=1}^N a_{nk}(S^k x)_\nu \rightarrow t_n^\nu$ as $N \rightarrow \infty$ uniformly for $\nu = 1, 2, \dots$ for each n . This implies, for an S -limit L , that

$$L(t_n) = \lim_{N \rightarrow \infty} \sum_{k=1}^N a_{nk} L(S^k x) = \lim_{N \rightarrow \infty} \sum_{k=1}^N a_{nk} L(x) = L(x) \sum_{k=1}^{\infty} a_{nk}$$

since L is continuous and S -invariant. Hence $\lim_{n \rightarrow \infty} L(t_n) = L(x)$ since A is regular.

Therefore, since $L(t_n)$ tends to 0 and to $L(x)$, $x \in \Theta(S)$, which is the desired result.

THEOREM 4. *The closure of $(S - I)[m]$ in $(m, \|\cdot\|)$ is $\Theta(S)$.*

PROOF. Since an S -limit L is S -invariant, clearly $(S - I)[m] \subset \Theta(S)$. We shall show that every positive linear functional which is zero on $(S - I)[m]$ is also zero on $\Theta(S)$. (The positive linear functionals are total in the dual of $\Theta(S)$; see, e.g. [1, Theorem 4, p. 217].)

Suppose q is a positive linear functional on $\Theta(S)$ which vanishes on $(S - I)[m]$. We can extend q to a positive linear functional G on m with $G(\bar{1}) = 1$ (see, e.g., [5, p. 20]). Also $G((S - I)x) = 0$ for all $x \in m$, so G is S -invariant. Hence G is an S -limit, so G is zero on $\Theta(S)$. Therefore the closure of $(S - I)[m]$ is $\Theta(S)$.

COROLLARY 1. $\overline{bs} = f_0$.

PROOF. $bs = \{x: \sup_n |\sum_{k=1}^n x_k| < \infty\}$. It is easy to show $bs = (T - I)[m]$ where T is the translation matrix.

f_0 is the space of sequences that are almost convergent to zero which is precisely $\Theta(T)$.

The following result links S -limits to $A \cdot \Sigma$ summability.

THEOREM 5. *If A is regular and if $A \cdot \Sigma$ is S -invariant, then $n_{A \cdot \Sigma} \cap m = \Theta(S)$.*

PROOF. $A \cdot \Sigma$ is S -invariant, so $(S - I)[m] \subseteq n_{A \cdot \Sigma} \cap m$ and $n_{A \cdot \Sigma} \cap m \subseteq \Theta(S)$ from Theorem 3. Since $n_{A \cdot \Sigma} \cap m$ is closed in m and $\overline{(S - I)[m]} = \Theta(S)$, we have $n_{A \cdot \Sigma} \cap m = \Theta(S)$.

COROLLARY. *If, in addition, $\bar{I} \in c_{A \cdot \Sigma}$, then $c_{A \cdot \Sigma} \cap m = \Lambda(S)$.*

We note that $\bar{I} \in c_{A \cdot \Sigma}$ if $A \cdot \Sigma$ is regular or if A is regular and S is positive with all rows adding up to 1. A is said to be compatible to S if A is regular, $A \cdot \Sigma$ is a regular, S -invariant sequence of matrices. Thus if A is compatible to S , $c_{A \cdot \Sigma} \cap m = \Lambda(S)$. Necessary and sufficient conditions for compatibility can be formulated using Silverman-Toeplitz and Shur Theorems for $\mathcal{Q}$ -summability found in [2].

Comparison theorems. We shall assume A is compatible to S . Let $\mathfrak{S} = A \cdot \Sigma$. Clearly if B were also compatible to S , then $c_{A \cdot \Sigma} \cap m = c_{B \cdot \Sigma} \cap m$. We show that among regular matrices the compatible ones are the "strongest".

THEOREM 6. *If B is regular, then $n_{B \cdot \Sigma} \cap m \subset n_{A \cdot \Sigma} \cap m$ if, in addition, $1 \in c_{B \cdot \Sigma}$, $c_{B \cdot \Sigma} \cap m \subset c_{A \cdot \Sigma} \cap m$.*

PROOF. Theorem 3 gives that $n_{B \cdot \Sigma} \cap m \subset \Theta(S)$ and compatibility gives that $n_{A \cdot \Sigma} \cap m = \Theta(S)$. This gives the desired conclusion.

If $1 \in c_{B \cdot \Sigma}$, since $A \cdot \Sigma$ is regular, we have $c_{B \cdot \Sigma} \cap m \subset c_{A \cdot \Sigma}$.

We say a matrix B is strongly $\mathcal{Q}$ -regular if $Bx \rightarrow l$ whenever $\mathcal{Q}x \rightarrow l$. For $\mathfrak{T} = C \cdot \Gamma$ where C is a Cesàro matrix and $\Gamma = \{T^\nu\}$ the sequence of powers of T , strongly $\mathfrak{T}$ -regular corresponds to strongly regular [6]. We have the following generalization of Lorentz' Theorem.

THEOREM 7. *B is strongly $\mathfrak{S}$ -regular on m if and only if B is regular and S -invariant (or $\lim_{n \rightarrow \infty} \sum |(BS)_{nk} - B_{nk}| = 0$).*

PROOF. Suppose B is $\mathfrak{S}$ -regular on m . Then

$$n_B \cap m \supset n_{A \cdot \Sigma} \cap m \supset (S - I)[m].$$

So B is S -invariant. $A \cdot \Sigma$ is regular, so $x \rightarrow l$ implies $\mathfrak{S}x \rightarrow l$, thus $Bx \rightarrow l$. B is thus regular.

Conversely, B is S -invariant implies $n_B \cap m \supset (S - I)[m]$. $n_B \cap m$ is closed, so $n_B \cap m \supset \Theta(S)$. B is regular, thus $c_B \cap m \supset \Lambda(S)$; that B and $\mathfrak{S}$ are consistent follows from $n_B \cap B \supset \Theta(S)$.

As a consequence, we have a shortened proof of Lorentz' characterization of strongly regular matrices [6, Theorem 7].

COROLLARY. *B is strongly regular if and only if B is regular and $\lim_{n \rightarrow \infty} \sum |b_{nk} - b_{n,k+1}| = 0$.*

PROOF. This is the above theorem applied to almost convergence, noting that $(BT)_{nk} = B_{n,k-1}$.

For $\mathcal{Q} = \{A^v\}$, consider the set $\mathcal{U}$ of all matrices U where the n th row of U is the n th row of A^v for some v . Following along the idea of G. M. Petersen [7], it can be shown [2] that $\mathcal{Q}$ sums exactly those sequences that are summed by every member of $\mathcal{U}$ to the same value.

The following generalizes a finding of R. J. Duran [4] for almost convergence.

THEOREM 8. $c_S \cap m = \cap \{c_B \cap m: B \text{ is regular and } S\text{-invariant}\}$.

PROOF. If B is regular and S -invariant, it is strongly S -regular on m , hence $c_S \cap m \subset c_B \cap m$. Also, $c_S = \cap \{c_U: U \in \mathcal{U}\}$ where $\mathcal{U}$ is described above. $c_S \subset c_U$ are consistent implies that each U is strongly S -regular and, hence, regular and S -invariant. Thus $\mathcal{U}$ is a subcollection of all regular and S -invariant matrices, so $\cap \{c_U: U \in \mathcal{U}\} \supset \cap \{c_B: B \text{ is regular and } S\text{-invariant}\}$, so $c_S \cap m = \cap \{c_B \cap m: B \text{ is regular and } S\text{-invariant}\}$.

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