UNIQUENESS THEOREMS FOR HARMONIC FUNCTIONS OF EXPONENTIAL GROWTH

DORON ZEILBERGER

ABSTRACT. Two uniqueness theorems for harmonic functions of exponential growth are proved. The first is a generalization to R^n of a theorem proved by Boas [1] for R^2 .

1. Introduction and statement of results. The purpose of this paper is to prove the following two theorems.

THEOREM A. *Let u be a real-valued harmonic function in R^n satisfying $|u(x)| < Ce^{A|x|}$, where $A < \pi$, $|x| = \sum_{i=1}^n |x_i|$ and C is a constant. If u vanishes on the integer lattice points of the hyperplanes*

$$x_n = 0, \quad \text{and} \quad x_n = a \quad (|a| \leq (1/(n-1))^{1/2}),$$

then it vanishes identically.

THEOREM B. *Let u be as above and suppose both u and $\partial u / \partial x_n$ vanish on the integer lattice points of $x_n = 0$; then u vanishes identically.*

Theorem A is a generalization of a theorem of Boas [1], who proved it for $n = 2$. Boas used the fact that in the two-dimensional case every real-valued entire harmonic function is the real part of an entire (analytic) function. Evidently, this method does not generalize to higher dimensions. Our strategy will be, instead, to view u as a "distribution" (i.e., a continuous linear functional) on the test space of bounded analytic functions on the polystrip $\times_{i=1}^n \{| \operatorname{Im} t_i | < A''\} \subset \mathbb{C}^n$ for $A'' > A$. The referee has kindly informed us that Rao [2] has proved Theorem A by different methods.

2. Proof of the results. We shall proceed by a sequence of lemmas.

LEMMA 1. *If u is harmonic in R^n and $|u(x)| < Ce^{A|x|}$, then any partial derivative of u enjoys the same properties.*

PROOF. For any $x_0 \in R^n$ look at the Poisson representation formula for the ball $\|y - x_0\| < 1$, differentiate under the integral sign and estimate.

Let $\mathcal{Q}$ be the class of analytic functions of n complex variables of the form

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$$\hat{v}(t) = (2\pi)^{n/2} \int v(x) e^{ixt} \quad \left(xt = \sum x_i t_i \right),$$

where $v \in C_0^\infty(R^n)$. All these functions are bounded in

$$K_{A''} = \bigcap_{i=1}^n \{ |\operatorname{Im} t_i| < A'' \} \quad \text{for every } A''.$$

Define a linear functional on $\mathcal{Q}$ by

$$(2.1) \quad T_u \left((2\pi)^{n/2} \int_{R^n} v(x) e^{ixt} \right) = \int_{R^n} u(x) v(x).$$

The next lemma will show that T_u can be extended continuously to $\mathcal{H}_{A''}$, the Banach space of bounded holomorphic functions on $K_{A''}$, provided $A'' > A$.

LEMMA 2. *Let u be harmonic in R^n and satisfy $|u(x)| < Ce^{A|x|}$. Let $A'' > A$; then T_u defined on $\mathcal{Q}$ by (2.1) can be extended to be a continuous linear functional on the Banach space $\mathcal{H}_{A''}$, which consists of bounded analytic functions on $K_{A''}$ where the norm is given by $\|f\|_{A''} = \sup_{t \in K_{A''}} |f(t)|$.*

PROOF. Let $A < A' < A''$ and let $R_+ = [0, \infty)$, $R_- = (-\infty, 0]$. Then

$$U_{\pm \pm \dots \pm}(t) = (2\pi)^{-n/2} \int_{R_{\pm} \times R_{\pm} \times \dots \times R_{\pm}} u(x) e^{-ixt} dx$$

belongs to $L^2(\times_{i=1}^n \{\operatorname{Im} t_i = \mp A'\})$, and for $\hat{v} \in \mathcal{Q}$,

$$(2.2) \quad \begin{aligned} T_u(\hat{v}) &= \sum_{\pm} \int_{R_{\pm} \times \dots \times R_{\pm}} u(x) v(x) \\ &= \sum_{\pm} \int_{\times_{i=1}^n \{\operatorname{Im} t_i = \mp A'\}} U_{\pm \dots \pm}(t) \hat{v}(t) dt_1 \dots dt_n. \end{aligned}$$

The sums in (2.2) each contain 2^n terms, corresponding to all possible choices of sign. Let us consider the term in the sum on the right-hand side of (2.2) involving $U_{- \dots -}(t)$, and let us write, for the moment, $\Omega = R_- \times R_- \times \dots \times R_-$. Then, by Green's formula,

$$(2.3) \quad \begin{aligned} (2\pi)^{n/2} U_{- \dots -}(t) &= \int_{\Omega} u(x) e^{-ixt} dx = \int_{\Omega} u(x) \Delta \left(\frac{-e^{-ixt}}{t_1^2 + \dots + t_n^2} \right) \\ &= - \int \Delta u(x) \frac{e^{-ixt}}{t_1^2 + \dots + t_n^2} + \int_{\partial\Omega} u(x) \frac{\partial}{\partial n} \left[\frac{-e^{-ixt}}{t_1^2 + \dots + t_n^2} \right] d\sigma \\ &\quad - \int_{\partial\Omega} \frac{\partial u}{\partial n} \frac{-e^{-ixt}}{t_1^2 + \dots + t_n^2} d\sigma. \end{aligned}$$

The first term on the right-hand side of (2.3) vanishes since u is harmonic. Now $\partial\Omega$ consists of n pieces: $\partial\Omega = \bigcup_{i=1}^n \{x_i = 0\} \cap \Omega$.

Let us consider the contribution from the face $x_1 = 0$. Here $\partial/\partial n = \partial/\partial x_1$ and

$$\begin{aligned}
 (2.4) \quad & \int_{\{x_1=0\} \cap \Omega} u(x) \frac{\partial}{\partial n} \left(\frac{-e^{-ixt}}{t_1^2 + \dots + t_n^2} \right) dx_2 \dots dx_n \\
 &= \int_{\{x_1=0\} \cap \Omega} u(0, x_2, \dots, x_n) \frac{it_1 e^{-ixt} d\sigma}{t_1^2 + \dots + t_n^2}
 \end{aligned}$$

and

$$\begin{aligned}
 (2.5) \quad & \int_{\{x_1=0\} \cap \Omega} \frac{\partial u}{\partial n}(x) \frac{e^{-ixt} d\sigma}{t_1^2 + \dots + t_n^2} \\
 &= \int_{\{x_1=0\} \cap \Omega} \frac{\partial u}{\partial x_1}(0, x_2, \dots, x_n) \\
 &\quad - \frac{it_1 \exp(-ix_2 t_2 - \dots - ix_n t_n)}{t_1^2 + \dots + t_n^2} dx_2 \dots dx_n.
 \end{aligned}$$

Now look at (2.2), the contribution from (2.4) is

$$\begin{aligned}
 & \int_{\{\operatorname{Im} t_i = A'\} \times \times_2^n \{\operatorname{Im} t_i = A'\}} \hat{v}(t) dt_1 \dots dt_n \\
 & \times \int_{\{x_1=0\} \cap \Omega} u(0, x_2, \dots, x_n) \frac{it_1 \exp(-ix_2 t_2 - \dots - ix_n t_n)}{t_1^2 + \dots + t_n^2} dx_2 \dots dx_n.
 \end{aligned}$$

But there is a similar contribution, with an opposite sign, from integration on $\{\operatorname{Im} t_1 = -A'\} \times \times_2^n \{\operatorname{Im} t_i = A'\}$. Let $\Gamma_{A'}$ be the rectangular contour in the t_1 -plane with sides $\pm iA' \pm R$; then, as $R \rightarrow \infty$, the sum of these contributions is

$$(2.6) \quad \int_{\times_2^n \{\operatorname{Im} t_i = A'\}} \phi(t_2, \dots, t_n) dt_2 \dots dt_n \int_{\Gamma_{A'}} \frac{\hat{v}(t_1, \dots, t_n) it_1}{t_1^2 + \dots + t_n^2} dt_1$$

where

$$\begin{aligned}
 \phi(t_2, \dots, t_n) &= \int_{\{x_1=0\} \cap \Omega} u(0, x_2, \dots, x_n) \\
 &\quad \times \exp(-ix_2 t_2 - \dots - ix_n t_n) dx_2 \dots dx_n.
 \end{aligned}$$

For fixed $t_2, \dots, t_n$,

$$\begin{aligned}
 & \int_{\Gamma_{A'}} \frac{\hat{v}(t_1, \dots, t_n) it_1}{t_1^2 + \dots + t_n^2} dt_1 \\
 &= \begin{cases} \pi i \{ \hat{v}(\tau_1, t_2, \dots, t_n) + \hat{v}(-\tau_1, t_2, \dots, t_n) \} & \text{if } |\operatorname{Im} \tau_1| < A', \\ 0 & \text{if } |\operatorname{Im} \tau_1| > A', \end{cases}
 \end{aligned}$$

where $\tau_1 = \tau_1(t_2, \dots, t_n)$ is given by

$$\tau_1^2 + t_2^2 + \dots + t_n^2 = 0$$

i.e., $\tau_1 = i(t_2^2 + \dots + t_n^2)^{1/2}$. Now

$$M_{A'} = \{(t_2, \dots, t_n) \in \mathbb{C}^{n-1}, \operatorname{Im} t_2 = A', \dots, \operatorname{Im} t_n = A', |\operatorname{Im} \tau_1| < A'\}$$

is seen to be a compact subset of $\times_{i=2}^n \{\operatorname{Im} t_i = +A'\}$, and we get that the contribution from the pair of boundary terms (obtained in (2.3)) considered is

$$(2.6') \quad \pi i \int_{M_{A'}} \phi(t_2, \dots, t_n) [\hat{v}(\tau_1, t_2, \dots, t_n) + \hat{v}(-\tau_1, t_2, \dots, t_n)] dt_2 \cdots dt_n,$$

and its absolute value is $\leq \text{constant } \|\hat{v}\|_{A''}$.

Similarly, if

$$\begin{aligned} \phi'(t_2, \dots, t_n) = & - \int_{\{x_1=0\} \cap R_- \times \cdots \times R_-} \frac{\partial u}{\partial x_1}(0, x_2, \dots, x_n) \\ & \times \exp(-ix_2 t_2 - \cdots - ix_n t_n) dx_2 \cdots dx_n, \end{aligned}$$

the net contribution from the two terms in (2.2) involving $\phi'(t_2, \dots, t_n)$ is

$$\int_{\{\operatorname{Im} t_2 = A'\} \times \cdots \times \{\operatorname{Im} t_n = A'\}} \phi'(t_2, \dots, t_n) \int_{\Gamma_{A'}} \frac{\hat{v}(t_1, \dots, t_n)}{t_1^2 + \cdots + t_n^2} dt_1$$

which is equal to

$$(2.7') \quad \begin{aligned} & \pi \int_{M_{A'}} \phi'(t_2, \dots, t_n) \\ & \cdot \frac{1}{\tau_1} [\hat{v}(\tau_1, t_2, \dots, t_n) - \hat{v}(-\tau_1, t_2, \dots, t_n)] dt_2 \cdots dt_n \end{aligned}$$

which, in absolute value, is $\leq \text{constant } \|v\|_{A''}$.

In a similar way we can consider all other terms of (2.2) and write it as a sum of $n2^{n-1}$ terms of the form (2.6') and $n2^{n-1}$ terms of the form (2.7'). The resulting formula defines $T_u(f)$ for every $f \in \mathcal{H}_{A''}$ and T_u is a bounded linear functional on $\mathcal{H}_{A''}$.

LEMMA. 3. For every $x \in R^n$, $T_u(e^{ixt}) = (2\pi)^{-n/2}u(x)$.

PROOF. Let K_ϵ be a C^∞ compact support approximate identity; then $\int K_\epsilon(y-x)e^{iyt} dt \rightarrow e^{ixt}$ in the topology of $\mathcal{H}_{A''}$ and

$$\begin{aligned} T_u(e^{ixt}) &= \lim_{\epsilon \rightarrow 0} T_u \left(\int K_\epsilon(y-x)e^{iyt} dy \right) \\ &= \lim_{\epsilon \rightarrow 0} (2\pi)^{-n/2} \int K_\epsilon(y-x)u(y) = (2\pi)^{-n/2}u(x). \end{aligned}$$

LEMMA 4. There exist measures $d\mu_1, d\mu_2$ on $\{t_n = 0\} = \mathbf{C}^{n-1}$ supported in the compact set

$$\begin{aligned} L_{A''} = \{ & (t_1, \dots, t_{n-1}); |\operatorname{Im} t_1| < A'', \dots, |\operatorname{Im} t_{n-1}| < A'', \\ & |\operatorname{Re}(t_1^2 + \cdots + t_{n-1}^2)^{1/2}| < A'' \} \end{aligned}$$

such that for every $f \in \mathcal{H}_{A''}$,

$$(2.8) \quad \begin{aligned} T_u(f) = & \int f(t_1, \dots, t_{n-1}, i(t_1^2 + \dots + t_{n-1}^2)^{1/2}) d\mu_1 \\ & + \int f(t_1, \dots, t_{n-1}, -i(t_1^2 + \dots + t_{n-1}^2)^{1/2}) d\mu_2. \end{aligned}$$

In particular,

$$(2.9) \quad \begin{aligned} u(x) = & \int \exp(ix_1 t_1 + \dots + ix_{n-1} t_{n-1}) \exp(-(t_1^2 + \dots + t_{n-1}^2)^{1/2} x_n) d\mu_1 \\ & + \int \exp(ix_1 t_1 + \dots + ix_{n-1} t_{n-1}) \exp((t_1^2 + \dots + t_{n-1}^2)^{1/2} x_n) d\mu_2. \end{aligned}$$

PROOF. Let $V_{A''} = \{(t_1, \dots, t_n) \in K_{A''}; t_1^2 + t_2^2 + \dots + t_n^2 = 0\}$. Then by the proof of Lemma 2, by adding all the terms like (2.6') and (2.7'), we get that there exists a measure $d\nu$, supported in $V_{A''}$ such that for every $f \in \mathfrak{H}_{A''}$, $T_u(f) = \int f d\nu$.

Let $d\nu = d\nu_1 + d\nu_2$, where $d\nu_1$ is supported in

$$\left\{ (t_1, t_2, \dots, i(t_1^2 + \dots + t_{n-1}^2)^{1/2}) \right\}$$

and $d\nu_2$ is supported in

$$\left\{ (t_1, t_2, \dots, t_{n-1}, -i(t_1^2 + \dots + t_{n-1}^2)^{1/2}) \right\}.$$

Let $d\mu_1, d\mu_2$ be the projections of $d\nu_1, d\nu_2$, respectively, on $t_n = 0$. Then the lemma follows since $d\mu_1, d\mu_2$ are supported in the projection of $V_{A''}$ on $t_n = 0$ which is $L_{A''}$.

Now we are in a position to prove the theorems.

PROOF OF THEOREM A. Since $A < \pi$ we can choose $A < A'' < \pi$. It is easily seen that $L_{A''}$ is contained in $\times_{i=1}^{n-1} \{|\operatorname{Im} t_i| < A''\} \times \{|\operatorname{Re} t_i| < A''\}$, and since $A'' < \pi$, the span of $\{e^{ixt}; x \in \mathbb{Z}^{n-1}\}$, where $\mathbb{Z}^{n-1}$ are the integer lattice points of $\mathbb{R}^{n-1}$, is dense in the space of bounded holomorphic functions on $L_{A''}$. By (2.9), $d\mu_1 + d\mu_2 \equiv 0$ and

$$\exp(-a(t_1^2 + \dots + t_{n-1}^2)^{1/2}) d\mu_1 + \exp(a(t_1^2 + \dots + t_{n-1}^2)^{1/2}) d\mu_2 \equiv 0.$$

Since $a \leq (1/(n-1))^{1/2}$ it follows that $d\mu_1, d\mu_2$ are supported in $\{t_1^2 + \dots + t_{n-1}^2 = 0\}$ and by (2.9), $u(x)$ is identically zero. $\square$

PROOF OF THEOREM B. Applying $\partial/\partial x_n$ to (2.9) we get

$$\begin{aligned} & \frac{\partial u}{\partial x_n}(x_1, \dots, x_{n-1}, 0) \\ &= - \int \exp(ix_1 t_1 + \dots + ix_{n-1} t_{n-1}) (t_1^2 + \dots + t_{n-1}^2)^{1/2} d\mu_1 \\ & \quad + \int \exp(ix_1 t_1 + \dots + ix_{n-1} t_{n-1}) (t_1^2 + \dots + t_{n-1}^2)^{1/2} d\mu_2. \end{aligned}$$

As in the proof of the Theorem A we get that

$$d\mu_1 + d\mu_2 \equiv 0, \quad (t_1^2 + \dots + t_{n-1}^2)^{1/2} (d\mu_1 - d\mu_2) \equiv 0.$$

Thus $d\mu_1 = -d\mu_2$ is supported at the set $\{t_1^2 + \dots + t_{n-1}^2 = 0\}$, and by

using (2.9) it once again follows that u vanishes identically.

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DEPARTMENT OF MATHEMATICS, WEIZMANN INSTITUTE OF SCIENCE, REHOVOT, ISRAEL