STATIONARY POINTS OF $(Z_2)^k$ -ACTIONS

FRANK L. CAPOBIANCO

ABSTRACT. This paper classifies up to bordism the manifolds with $(Z_2)^k$ -action fixing $RP(n)$, $CP(n)$, or S^n .

1. Introduction. In [2], Conner and Floyd showed that the fixed point structure of a differentiable involution on a closed manifold determines the bordism class. They initiated several specific applications of this fact, some of which were completed elsewhere (e.g. [1], [3], [5]). In [4], Stong showed more generally that the stationary point structure of a differentiable $(Z_2)^k$ -action determines the bordism class. In this paper we note several applications of this latter fact, among them

PROPOSITION 3.5. *If (V^s, T) is a differentiable $(Z_2)^k$ -action on a closed manifold whose set of stationary points is $RP(2n)$, there is an integer j , $0 \leq j \leq k$, such that $s = n \cdot 2^{j+1}$ and (V^s, T) is bordant to $((RP(2n))^{2^j}, T_j)$ where T_j is the twist.*

I am indebted to Professor R. E. Stong both for suggesting this topic to me and much more so for advising me throughout my graduate education.

2. $(Z_2)^k$ -actions. We begin with some general concepts, recalling the notation and results of Stong [4]:

We regard $(Z_2)^k$ as being generated by the elements $t_1, \dots, t_k$ subject to the relations $t_i^2 = 1$ and $t_i t_j = t_j t_i$. $\text{Hom}((Z_2)^k, Z_2)$, the set of homomorphisms $(Z_2)^k \rightarrow Z_2 = \{+1, -1\}$, consists of 2^k distinct homomorphisms which we label f_i , $i = 1, \dots, 2^k$. We agree to let $f_1 = 1$. Thus if $2 \leq i \leq 2^k$, there is a generator t_j in $(Z_2)^k$ such that $f_i(t_j) = -1$. We note for $2 \leq i \leq 2^k$, $\ker f_i$ is isomorphic to $(Z_2)^{k-1}$.

Every irreducible representation of $(Z_2)^k$ is one-dimensional and has the form $\lambda: (Z_2)^k \times R \rightarrow R$ with $\lambda(t, x) = f_i(t) \cdot x$ for some f_i . Hence if $\mathcal{E}$ is a real n -plane bundle with $(Z_2)^k$ -action covering a compact Hausdorff space with $(Z_2)^k$ -action, there are 2^k subbundles $\mathcal{E}_i$ of $\mathcal{E}$ with $\mathcal{E} = \mathcal{E}_1 \oplus \dots \oplus \mathcal{E}_{2^k}$ such that the action of t_j on $\mathcal{E}$ corresponds to $(f_1(t_j)x_1, \dots, f_{2^k}(t_j)x_{2^k})$ where $x_i \in E(\mathcal{E}_i)$, $i = 1, \dots, 2^k$. We shall henceforth refer to this as the decomposed form of $\mathcal{E}$.

Let (V^s, T) be a closed manifold with differentiable $(Z_2)^k$ -action. $x \in V$ is a stationary point of T if $T(t, x) = x$ for all $t \in (Z_2)^k$. The set of stationary points N of T is a disjoint union of closed manifolds. If $\mathcal{E} \rightarrow N$ represents the normal bundle of N in V^s , then $\mathcal{E}$ decomposes as above. Since $f_1 = 1$, $\mathcal{E}_1$

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is the trivial zero-dimensional bundle and may be ignored. Let N^n be an n -dimensional component of N and for $2 \leq i \leq 2^k$ let N_i represent the set of stationary points of $\ker f_i$. If $t_j \notin \ker f_i$, then $\mathcal{E}_i \rightarrow N^n$ is a component of the normal bundle to the fixed point set of the closed manifold with involution $(N_i^m, T(t_j, -)|N_i^m)$, where N_i^m represents the component of N_i containing N^n .

The key fact from [4] that we use is that there is a monomorphism

$$F : \mathfrak{N}_*((Z_2)^k, 0) \rightarrow \sum_{(s_2, \dots, s_{2^k})} \mathfrak{N}_*(BO(s_2) \times \cdots \times BO(s_{2^k}))$$

where $\mathfrak{N}_*((Z_2)^k, 0)$ is the bordism module of differentiable $(Z_2)^k$ -actions on closed manifolds. F is defined by sending $[V^s, T]$ in $\mathfrak{N}_*((Z_2)^k, 0)$ to $[\mathcal{E}_2 \oplus \cdots \oplus \mathcal{E}_{2^k} \rightarrow N]$, where $\mathcal{E}_2 \oplus \cdots \oplus \mathcal{E}_{2^k} \rightarrow N$ is the decomposed form of the normal bundle to the set of stationary points of (V^s, T) .

3. Applications.

PROPOSITION 3.1. *If (V^s, T) is a differentiable $(Z_2)^k$ -action on a closed manifold and N is the set of stationary points of T , then $\chi(V^s) = \chi(N) \pmod{2}$ where $\chi(\)$ denotes the Euler characteristic.*

PROOF. If $k=1$, this is [2, 27.2]. Since N is the fixed point set of the manifold with involution $(N_i, T(t_j, -)|N_i)$ as outlined in §2, the result follows by induction.

We label an arbitrary element of R^{2^k-1} as $(x_2, \dots, x_{2^k})$. We define a $(Z_2)^k$ -action on R^{2^k-1} by $t_i \cdot (x_2, \dots, x_{2^k}) = (f_2(t_i)x_2, \dots, f_{2^k}(t_i)x_{2^k})$ for $1 \leq i \leq k$. This action induces a $(Z_2)^k$ -action T on $RP(2^k-2)$ such that the set of stationary points of $(RP(2^k-2), T)$ consists of 2^k-1 isolated points; hence, the set of stationary points of $(RP(2^k-2), T)^m$ consists of $(2^k-1)^m$ isolated points. This example is very useful: for instance, it shows that [2, §27] and [1] do not extend directly to $(Z_2)^k$ -actions. (One may consider [6] in this regard also.)

PROPOSITION 3.2. *If (V^s, T) is a differentiable $(Z_2)^k$ -action on a closed manifold whose set of stationary points is $RP(2n+1)$, then (V^s, T) bords.*

PROOF. Suppose $\mathcal{E}_2 \oplus \cdots \oplus \mathcal{E}_{2^k} \rightarrow RP(2n+1)$ is the decomposed form of the normal bundle to $RP(2n+1)$ in V^s . As outlined in §2, for each $2 \leq i \leq 2^k$, $\mathcal{E}_i \rightarrow RP(2n+1)$ is the fixed point set of a closed manifold with involution. An argument similar to that of [2, 27.7] shows that for each i , $w(\mathcal{E}_i) = (1+a)^{2^p}$ where a is the one-dimensional generator of the Z_2 -cohomology of $RP(2n+1)$. Hence

$$[\mathcal{E}_2 \oplus \cdots \oplus \mathcal{E}_{2^k} \rightarrow RP(2n+1)] = 0$$

in

$$\sum_{(s_2, \dots, s_{2^k})} \mathfrak{N}_*(BO(s_2) \times \cdots \times BO(s_{2^k}))$$

and the result follows from the fact F is monic.

We remark that the analogous proposition is true and proved similarly for $(Z_2)^k$ -actions fixing $CP(2n+1)$.

PROPOSITION 3.3. *If (V^s, T) is a differentiable $(Z_2)^k$ -action on a closed manifold whose set of stationary points is S^n , then (V^s, T) bords.*

PROOF. Suppose $\mathcal{E}_2 \oplus \cdots \oplus \mathcal{E}_{2^k} \rightarrow S^n$ is the decomposed form of the normal bundle to S^n in V^s . As outlined in §2, for each $2 \leq i \leq 2^k$, $\mathcal{E}_i \rightarrow S^n$ is the fixed point set of a closed manifold with involution. It follows from [2, 25.3] that $w(\mathcal{E}_i) = 1$. Hence $[\mathcal{E}_2 \oplus \cdots \oplus \mathcal{E}_{2^k} \rightarrow S^n] = 0$ and the result follows.

PROPOSITION 3.4. *If M^n bords, then for each $p \geq 0$ there is a closed manifold V^{n+p} with $(Z_2)^k$ -action T fixing M^n . Furthermore, this manifold with $(Z_2)^k$ -action bords.*

PROOF. If $p = 0$, let $(V^n, T) = (M^n, 1)$. If $p \geq 1$, suppose $M^n = d(B^{n+1})$ and let D^p and S^{p-1} represent the p -disk and the $(p-1)$ -sphere, respectively. Let A be the $(Z_2)^k$ -action on D^p defined by $A(t_j, x) = -x$ for $j = 1$, and $A(t_j, x) = x$ for $2 \leq j \leq k$. Let

$$V^{n+p} = \frac{M^n \times D^p \cup B^{n+1} \times S^{p-1}}{M^n \times S^{p-1} \equiv M^n \times S^{p-1}}$$

and $T = 1 \times A \cup 1 \times A$. Since the decomposed form of the normal bundle to M^n in V^{n+p} is trivial, the second assertion follows.

We consider a space X and the cartesian product X^{2^k} of 2^k copies of X . We define an action T of $(Z_2)^k$ on X^{2^k} , which we call the twist, as follows: Let t_j be a generator of $(Z_2)^k$. Identify X^{2^k} with $\times_{i=1}^{2^{k-1}} (Y_i \times Y'_i)$ where $Y_i = Y'_i = X^{2^{k-j}}$ by identifying $(x_1, \dots, x_{2^k})$ with $((y_1, y'_1), \dots, (y_{2^{j-1}}, y'_{2^{j-1}}))$ where

$$y_1 = (x_1, \dots, x_{2^{k-j}}), \quad y'_1 = (x_{2^{k-j}+1}, \dots, x_{2^{k-j+1}}), \dots, \\ y'_{2^{j-1}} = (x_{2^{k-2^{j-1}}+1}, \dots, x_{2^k}).$$

Define the action of t_j on X^{2^k} by sending $((y_1, y'_1), \dots, (y_{2^{j-1}}, y'_{2^{j-1}}))$ to $((y'_1, y_1), \dots, (y'_{2^{j-1}}, y_{2^{j-1}}))$. The set of stationary points of (X^{2^k}, T) is X .

We consider the following $(Z_2)^k$ -action on $(RP(2n))^{2^j}$ which we label T_j . Write $(Z_2)^k = (Z_2)^j \times (Z_2)^{k-j}$. Let T_j be the twist action of $(Z_2)^j$ and the trivial action of $(Z_2)^{k-j}$.

PROPOSITION 3.5. *If (V^s, T) is a differentiable $(Z_2)^k$ -action on a closed manifold whose set of stationary points is $RP(2n)$, then there is an integer j , $0 \leq j \leq k$, such that $s = n \cdot 2^{j+1}$ and (V^s, T) is bordant to $((RP(2n))^{2^j}, T_j)$.*

PROOF. Let F^{2n} be a connected closed manifold of odd Euler characteristic. Suppose (V^s, T) is a differentiable $(Z_2)^k$ -action on a closed manifold whose set of stationary points is F^{2n} . Let

$$\mathcal{E}_2^{s_2} \oplus \cdots \oplus \mathcal{E}_{2^k}^{s_k} \rightarrow F^{2n}$$

be the decomposed form of the normal bundle to F^{2n} in V^s , where $s_i = \dim \mathcal{E}_i$. Let N_i , for $2 \leq i \leq 2^k$, represent the set of stationary points of $\ker f_i \cong (Z_2)^{k-1}$. Let

$$\nu_2^{s_2} \oplus \cdots \oplus \nu_{2^k}^{s_k-1} \rightarrow N_i$$

be the decomposed form of the normal bundle to the set of stationary points of $\ker f_i$. Let N_{i_0} be the component of N_i which contains F^{2n} . Note that if

$t_j \notin \ker f_i$, $T(t_j, -)|_{N_i}$ is an involution whose fixed set on N_{i_0} is F^{2n} and which is free off N_{i_0} . Since every component of N_i except N_{i_0} must therefore bord, $\chi(F^{2n}) = \chi(N_{i_0}) = 1 \bmod 2$. (Since $\mathcal{E}_i^s \rightarrow F^{2n}$ is the normal bundle to the fixed set of $(N_{i_0}, T(t_j, -)|_{N_{i_0}})$, we mention as an aside that by [2, 27.5], $w_{s_i}(\mathcal{E}_i) \neq 0$.)

We construct a similar argument for N_{i_0} : For each j , $2 \leq j \leq 2^{k-1}$, let $g_j \in \text{Hom}(\ker f_i, Z_2)$ and let M_j be the set of stationary points of the action of $\ker g_j$ on V^s . Let M_{j_0} be the component of M_j containing N_{i_0} . Note that if $r \notin \ker g_j$, $v_j^p \rightarrow N_{i_0}$ is the normal bundle to a component of the fixed point set of the manifold with involution $(M_{j_0}, T(r, -)|_{M_{j_0}})$. Since N_{i_0} is the only nonbounding component of N_i , $\chi(M_{j_0}) = \chi(N_{i_0}) = 1 \bmod 2$. It follows from [6] and [2, 27.5] that $w_{p_j}(v_j) \neq 0$.

Suppose $F^{2n} = RP(2n)$. By [2, 27.7] and [5], if $s_i \neq 0$ then $s_i = 2n$ and $w(\mathcal{E}_i) = (1 + a)^{2n+1}$ where a is the one-dimensional generator of $H^*(RP(2n); Z_2)$. If the number of s_i such that $s_i \neq 0$ is precisely $2^j - 1$, $0 \leq j \leq k$, then $s = n \cdot 2^{j+1}$ and (V^s, T) is bordant to $((RP(2n))^{2^j}, T_j)$ from the fact F is monic. If the number of s_i such that $s_i \neq 0$ is not precisely $2^j - 1$, there is an integer m , $2 \leq m \leq 2^{k-1}$ such that $\bigvee_m^{p_m} RP(2n) = \mathcal{E}_a^{2n} \oplus \mathcal{E}_b^{2n}$ while $\dim N_{i_0} = 2n$. Since $p_m = 4n$ and $w_{4n}(\bigvee_m) \neq 0$, we see this latter case cannot occur.

We remark that the analogous proposition is true and proved similarly for $(Z_2)^k$ -actions fixing $CP(2n)$.

As an immediate corollary to the proof of 3.4 we have the following generalization of [2, 27.5]:

PROPOSITION 3.6. *Let F^{2n} be a connected closed manifold of odd Euler characteristic. Suppose (V^{2s}, T) is a closed manifold with differentiable $(Z_2)^k$ -action whose set of stationary points is F^{2n} . If*

$$\mathcal{E}_2^{s_2} \oplus \cdots \oplus \mathcal{E}_{2^k}^{s_{2^k}} \rightarrow F^{2n}$$

is the decomposed form of the normal bundle, then $w_{s_i}(\mathcal{E}_i) \neq 0$ for each i , $2 \leq i \leq 2^k$.

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