THE PREPARATION THEOREM AND THE FREENESS OF $A[[X]]/I$

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ABSTRACT. Let I be a nonzero ideal of $A[[X]]$, the ring of formal power series over a commutative Noetherian ring A . These are equivalent: (i) I is generated by a single series $f = a_0 + a_1X + \dots$ such that for some s , a_s is a unit, the first s coefficients $a_0, \dots, a_{s-1}$ of f lie in the Jacobson radical of A and A is complete in the adic topology defined by the ideal they generate. (ii) $A[[X]]/I$ is a free A -module.

A preliminary result from Bourbaki's presentation [1, p. 38, Proposition 5] of the Preparation Theorem asserts (essentially) that $A[[X]]/fA[[X]]$ is a finitely generated free A -module if A is a local complete Hausdorff commutative ring and some coefficient of f is a unit. A generalization of this result is obtained by shifting the hypothesis from A to f . For a Noetherian ring the converse to the generalization is true as well.

By definition, a formal power series $f = a_0 + a_1X + \dots$ in one variable over a commutative ring A is of reduced order s if $a_0, \dots, a_{s-1}$ belong to the Jacobson radical $\text{rad}(A)$ of A and a_s is a unit in A . A polynomial of degree s is said to be distinguished if it is monic and of reduced order s . Only commutative rings with unit, unital modules, etc. are considered here.

THEOREM 1 [1, p. 38, PROPOSITION 5]. *Let $f = a_0 + a_1X + \dots$ be a formal power series of reduced order s over a ring A . Let $\mathfrak{m} = (a_0, \dots, a_{s-1})$ and let $M = A + AX + \dots + AX^{s-1}$ with the understanding that $\mathfrak{m}$ and M are zero if $s = 0$. If A is complete Hausdorff in the $\mathfrak{m}$ -adic topology, then $A[[X]] = fA[[X]] + M$ and $fA[[X]] \cap M = 0$.*

The proof of Theorem 1 goes exactly as in [1, p. 38] where it is assumed that A is local complete Hausdorff in its maximal ideal adic topology.

COROLLARY 2. *If $f = a_0 + a_1X + \dots$ is a formal power series of reduced order s over a ring A such that A is complete and Hausdorff in the $(a_0, \dots, a_{s-1})$ -adic topology, then $A[[X]]/fA[[X]]$ is a free A -module of rank s . In fact, the images of the first s powers of X form a basis of $A[[X]]/fA[[X]]$.*

COROLLARY 3 (THE PREPARATION THEOREM). *If $f = a_0 + a_1X + \dots$ is a formal power series of reduced order s over a ring A such that A is complete and separated in the $(a_0, \dots, a_{s-1})$ -adic topology, then there exists a unique pair*

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(u, F) such that u is invertible in $A[[X]]$, F is a distinguished polynomial of degree s and $f = uF$.

The first corollary is just a restatement of the theorem. The proof of the second uses the theorem in the same way as [1, p. 41, Proposition 6]. We now come to the point of this paper: the converse to Theorem 1. The proof requires that the situation be preserved after a base change of the form $A \rightarrow A/n$. In general this does not happen since

$$A[[X]] \otimes (A/n) \rightarrow (A/n)[[X]]$$

is not bijective unless n is finitely generated. So the Noetherian hypothesis is forced on us. The following was inspired by the work of A. Viola-Prioli [2] and was discovered before its relationship to the Preparation Theorem.

THEOREM 4. *Let A be a Noetherian ring and I a nonzero ideal of $A[[X]]$, the ring of formal power series over A in one variable. If $A[[X]]/I$ is a projective A -module of constant finite rank s , then I is generated by one element $f = a_0 + a_1X + \dots$ of reduced order s such that A is complete and Hausdorff in the $(a_0, \dots, a_{s-1})$ -adic topology.*

PROOF. Let $B = A[[X]]$ and $C = B/I$. The first step in proving I cyclic is to apply the base change $A \rightarrow A/\text{rad}(A)$. Since $\text{rad}(A)$ is finitely generated the situation remains unchanged except that the new ring of scalars has zero radical. To save putting bars over everything in sight, suppose for a moment that A has zero radical. Let $\mathfrak{c}$ be the ideal of A generated by the coefficients of elements of I . Let $\mathfrak{c}_j$ be the ideal of coefficients of terms of degree j of elements of I . Then $\mathfrak{c}_0 \subseteq \mathfrak{c}_1 \subseteq \dots$ and $\bigcup \mathfrak{c}_j = \mathfrak{c}$. We will see that $\mathfrak{c} = A$. To see this, apply the change of scalars $A \rightarrow \bar{A} = A/\mathfrak{c}$. Since $I \subseteq \mathfrak{c}B$ and I is A -pure in B , $\mathfrak{c}I = \mathfrak{c}B \cap I = I$. Thus $I \otimes \bar{A} = I/\mathfrak{c}I = 0$ and so $\bar{A} \otimes B \rightarrow \bar{A} \otimes C$ is an $\bar{A}$ -algebra isomorphism. But since C is A -finite, this implies $\bar{A}[[X]] = \bar{A} \otimes B$ is $\bar{A}$ -finite which is impossible unless $\bar{A} = 0$. Thus $\mathfrak{c} = A$ and so there is a least integer n such that $\mathfrak{c}_n = A$. If $n = 0$ then I contains a unit of B so $C = 0$. The other part of the conclusion says A is complete and separated in the 0-adic (= discrete) topology which is always true. So we may assume that n is positive. Let $\bar{A} = A/\mathfrak{c}_{n-1}$ and let $u : B \rightarrow \bar{A}[[X]] = B \otimes \bar{A} = \bar{B}$ be the A -algebra homomorphism which reduces coefficients modulo $\mathfrak{c}_{n-1}$. We will see that $n = s$ by comparing ranks after showing that $I\bar{B} = X^n\bar{B}$. For then we will have

$$s = \text{rank}_A(C) = \text{rank}_{\bar{A}}(C \otimes \bar{A}) = \text{rank}_{\bar{A}}(\bar{B}/X^n\bar{B}) = n.$$

By the choice of n , there is an element g of I whose n th coefficient is 1. Write $g = h + X^n w$, where h is a polynomial of degree less than n and w is a unit in B . Evidently all the coefficients of h lie in $\mathfrak{c}_{n-1}$ since they are term by term equal to the first n coefficients of g and $\bar{g}$ is in I . Therefore $u(h) = 0$ and so $X^n = u(g/w)$ is an element of $u(I) = I\bar{B}$. On the other hand u kills the first n coefficients of each element of I , so clearly $u(I) = X^n\bar{B}$. This establishes that $s = n$. Next we remove the condition that $\text{rad}(A) = 0$. By the preceding argument there is an element f of I which is congruent to X^s modulo $\text{rad}(A)$ and for which $fB + \text{rad}(A)I = I$. Since B is Noetherian [3, p. 45, Corollary

6], I is B -finitely generated. Also $\text{rad}(A) \subseteq \text{rad}(B)$, so Nakayama's Lemma applies and we have that $fB = I$ and f is reduced of order s since it is congruent to X^s modulo $\text{rad}(A)$. It remains to show that A is complete and Hausdorff in the $\mathfrak{m} = (a_0, \dots, a_{s-1})$ -adic topology where $f = a_0 + a_1X + \dots$ is a generator of I of reduced order s . Since $\mathfrak{m} \subseteq \text{rad}(A)$ then A is Hausdorff in its $\mathfrak{m}$ -adic topology [3, p. 65, Corollary]. Let t be the image of X in C . The tC -adic topology and the XB -adic topology that C inherits from B coincide, so C is tC -adically complete. Because $a_0 = -t(a_1 + a_2t + \dots)$ and X belongs to $\text{rad}(B)$, we have $a_0C \subseteq tC \subseteq \text{rad}(C)$. Now [3, p. 119, Example 12] implies that C is also a_0C -adically complete. Let $A \rightarrow \hat{A}$ denote the a_0A -adic completion of A . It is faithfully flat [3, p. 72, Proposition 9] and so $A \rightarrow \hat{A}$ is injective. As C is A -projective of constant nonzero rank it is faithfully flat too. Thus the composite $A \rightarrow \hat{A} \rightarrow \hat{A} \otimes C$ is injective. By [3, p. 68, Theorem 3] $\hat{A} \otimes C$ is C -algebra isomorphic to the completion of C (with the a_0A -adic topology) which is just C since C is a_0C -adically ($= a_0A$ -adically) complete. Thus $\hat{A}$ is a finitely generated A -module since it is a submodule of one. Hence A is a_0A -adically complete by [3, p. 118, Exercise 9]. We finish the argument by induction on s . The induction hypothesis is this: If A' is a Noetherian ring, if $C' = B'/gB'$ is A' -projective of rank $s - 1$ where $g = b_0 + b_1X + \dots$ is reduced of order $s - 1$ and $B' = A'[[X]]$ then A' is complete in the $(b_0, \dots, b_{s-2})$ -adic topology. Let $A' = A/a_0A$ and write $f = a_0 + gX$. Now let g' be the image of g in $B' = A'[[X]]$ and let $C' = B'/g'B'$. Note that g' is reduced of order $s - 1$ since g is and $A \rightarrow A'$ carries $\text{rad}(A)$ into $\text{rad}(A')$. As $fB' \subseteq g'B'$ then there is an induced A' -algebra surjection $C \otimes A' \rightarrow C'$; hence C' is A' -finitely generated. For every prime $\mathfrak{p}'$ of A' , the image of g' in $B'/\mathfrak{p}'B'$ is a nonzero divisor because it is a nonzero element of an integral domain. Now [2, Proposition 1.1] implies that the A' -homomorphism $B' \rightarrow B'$ defined by multiplication by g' is A' -pure. Hence its cokernel C' is A' -flat. Thus C' is A' -projective since it is flat and finitely generated. The rank of C' is constantly $s - 1$ at the maximal ideals of A' since the first $s - 1$ coefficients of g' are in $\text{rad}(A')$. Hence C' has constant rank $s - 1$ on $\text{spec}(A')$. The induction hypothesis applies and we find that A/a_0A is $\mathfrak{m}/a_0A$ -adically complete since the ideal generated by the first $s - 2$ coefficients of g' is precisely $\mathfrak{m}/a_0A$. The following proposition implies that A is $\mathfrak{m}$ -adically complete which establishes the theorem.

PROPOSITION 5. *Let $\mathfrak{n} \subseteq \mathfrak{m}$ be ideals contained in the Jacobson radical of a Noetherian ring A . If A is $\mathfrak{n}$ -adically complete and if $A/\mathfrak{n}$ is $\mathfrak{m}/\mathfrak{n}$ -adically complete, then A is $\mathfrak{m}$ -adically complete.*

PROOF. First consider the case $\mathfrak{n}^2 = 0$. In this case $\mathfrak{n} = \mathfrak{n}/\mathfrak{n}^2 = \mathfrak{n} \otimes (A/\mathfrak{n})$ is $\mathfrak{m}/\mathfrak{n}$ -adically complete [3, p. 52, Corollary 1]. But on $\mathfrak{n}$ the $\mathfrak{m}/\mathfrak{n}$ -adic topology from $A/\mathfrak{n}$ and the $\mathfrak{m}$ -adic topology from A coincide since $(\mathfrak{m}/\mathfrak{n})^k \mathfrak{n} = \mathfrak{m}^k \mathfrak{n}$ for all k . Hence $\mathfrak{n}$ is $\mathfrak{m}$ -adically complete. By [3, p. 64, Theorem 2] the $\mathfrak{m}$ -adic topology on $\mathfrak{n}$ and its subspace topology are equivalent. So $\mathfrak{n}$ is a complete subspace of A relative to the $\mathfrak{m}$ -adic topology. Let (r_i) be an $\mathfrak{m}$ -adically Cauchy sequence in A . We pass to $A/\mathfrak{n}$ (which is $\mathfrak{m}$ -complete by hypothesis) and pull back a limit point r to A to find that for each k there is an i_k such that $r_i - r$ is in $\mathfrak{m}^k + \mathfrak{n}$ for all $i \geq i_k$. So we can construct a

sequence b_k of elements of $\mathfrak{n}$ such that for each k , $d_k = b_k - r_{i_k} + r$ lies in $\mathfrak{m}^k$. The sequence d_k converges $\mathfrak{m}$ -adically to zero and so is Cauchy. Therefore $(b_k) = (d_k) - (r - r_{i_k})$ is the difference of two $\mathfrak{m}$ -adically Cauchy sequences. Since $\mathfrak{n}$ is $\mathfrak{m}$ -adically complete there is an element b in $\mathfrak{n}$ such that $\lim b_k = b$. It follows that the subsequence (r_{i_k}) and hence the sequence (r_i) converge to $b + r$. Thus A is $\mathfrak{m}$ -adically complete. The argument for the case where $\mathfrak{n}$ is nilpotent is a straightforward induction on $k =$ the least integer p for which $\mathfrak{n}^p = 0$. It is omitted. Now let $u : A \rightarrow \hat{A}$ be the $\mathfrak{m}$ -adic completion of A . Both A and $\hat{A}$ are $\mathfrak{n}$ -adically complete. Let $u_k : A_k \rightarrow A \otimes A_k$ be the homomorphism obtained from u by tensoring with $A \rightarrow A/\mathfrak{n}^k$. Then since A and $\hat{A}$ are $\mathfrak{n}$ -adically complete, $u = \varprojlim u_k$ [3, p. 32]. On the other hand the nilpotent version of the proposition implies that each A_k is $\mathfrak{m}$ -adically complete and so u_k is an isomorphism for all k [3, p. 68, Theorem 3]. As the limit of isomorphisms is an isomorphism we see that $A \rightarrow \hat{A}$ is an isomorphism, hence A is $\mathfrak{m}$ -adically complete.

THEOREM 6. *Let A be a Noetherian ring and I a nonzero ideal of the ring of formal power series in one variable $A[[X]]$. These are equivalent:*

- (i) $A[[X]]/I$ is a free A -module.
 - (ii) $A[[X]]/I$ is a projective A -module of finite constant rank s .
 - (iii) I is generated by a power series $f = a_0 + a_1X + a_2X^2 + \dots$ of reduced order s such that A is complete in the $(a_0, \dots, a_{s-1})$ -adic topology.
 - (iv) I is generated by a distinguished polynomial $F = a_0 + a_1X^1 + \dots + X^s$ of degree s such that A is complete in the $(a_0, \dots, a_{s-1})$ -adic topology.
- Moreover when these conditions hold, $A[[X]]/I$ is generated by the first s powers of the image of X .

PROOF. (ii) implies (iii) by Theorem 4. (iii) implies (iv) by the Preparation Theorem. (iv) implies (iii) is obvious. Corollary 2 shows that (iii) implies (i) and establishes the last sentence of the theorem. It remains to prove that (i) implies (ii) and for this it is enough to show that $C = A[[X]]/I$ is A -finitely generated. Since A is Noetherian, $A[[X]]$ is a content A -module (see [4] for the definition). Since I is a pure submodule of $A[[X]]$, I is also a content A -module [4, 1.3]. Hence if $\mathfrak{p}I = I$ for each maximal ideal $\mathfrak{p}$ of A , then $\text{rad}(A)I = I$. But then Nakayama's Lemma implies $I = 0$, a contradiction. So we can find a maximal ideal $\mathfrak{p}$ of A such that $\mathfrak{p}I \neq I$. Apply the change of rings $A \rightarrow A/\mathfrak{p} = k$. $Ik[[X]]$ is nonzero by the choice of $\mathfrak{p}$ so $C \otimes k = k[[X]]/Ik[[X]]$ is a finite dimensional k -vector space. By the invariance of the rank of a free module C is A -finite.

Now we characterize the finite analytic extensions of a ring which are also projective modules.

COROLLARY 1. *Let A be a Noetherian ring and I a nonzero ideal of $A[[X]]$. These are equivalent:*

- (i) $A[[X]]/I$ is a finitely generated projective A -module.
- (ii) I is generated by one element f such that for some g in $A[[X]]$ the product $fg = a_0 + a_1X + \dots$ is a distinguished polynomial of degree s and A is complete in the $(a_0, \dots, a_{s-1})$ -adic topology.
- (iii) I is generated by one element f such that for some g in $A[[X]]$ the product

$fg = a_0 + a_1X + \dots$ is reduced of order s and A is complete in the $(a_0, \dots, a_{s-1})$ -adic topology.

PROOF. (ii) implies (iii) because a distinguished polynomial of degree s is reduced of order s . Let $B = A[[X]]$ and $C = B/I$. Assume (iii). Let $D = B/fgB$. Since fg lies in I there is an induced surjection of A -algebras $D \rightarrow C$. By Theorem 5, D is a finitely generated free A -module. Hence C is A -finite. Since fg is of reduced order s , then as in the proof of Theorem 4, the mapping $B \rightarrow B$ defined by multiplication by fg is a pure A -monomorphism. Therefore $C = B/fB$ is A -flat. Thus C is A -projective since it is flat and finitely generated. Therefore (iii) implies (i). Assume (i). Let $S = \text{spec}(A)$ and $S(n)$ the set of points $\mathfrak{p}$ of S such that $\text{rank}_{A_{\mathfrak{p}}}(C \otimes A_{\mathfrak{p}}) = n$. Since C is projective, its rank function is locally constant on S so the $S(n)$ are open sets in the Zariski topology [5, p. 138, Theorem 1]. Clearly the $S(n)$ are pairwise disjoint so we obtain by compactness a decomposition of S into pairwise disjoint open sets $S_1, \dots, S_t$ such that the rank of C is constant on S_j for each j . This partition of S gives in a natural way orthogonal idempotents $e_1, \dots, e_t$ of A such that for each j , $S_j = V(e_j)$ [5, p. 130, Proposition 15]. Letting $A_j = A/e_jA$ and applying the base change $A \rightarrow A_j$ produces a projective A_j -module $C \otimes A_j = A_j[[X]]/IA_j[[X]]$. Moreover the rank of $C \otimes A_j$ is constant on $S_j = \text{spec}(A_j)$. So by Theorem 5, $IA_j[[X]]$ is generated by a distinguished polynomial $F_j = a_{j0} + a_{j1}X + \dots + X^{r_j}$ where r_j is the rank of C on S_j and A_j is complete in the $\mathfrak{m}_j = (a_{j0}, \dots, a_{j_{r_j-1}})$ -adic topology. We are allowed to consider that A is the product $A_1 \times \dots \times A_t$. This induces a corresponding decomposition of the power series ring $A[[X]] = A_1[[X]] \times \dots \times A_t[[X]]$. Let $r = \max(r_j)$ and $H = F_1X^{r-r_1} + \dots + F_tX^{r-r_t}$. Let $F = F_1 + \dots + F_t$. It is clear that F generates I , that H is in I and that H is monic of degree r . The ideal generated by the first $r-1$ coefficients of H is equal to $\mathfrak{m} = \mathfrak{m}_1 \times \dots \times \mathfrak{m}_t$. That $\mathfrak{m}_j$ -adic completeness in each factor implies $\mathfrak{m}$ -adic completeness of the product is straightforward. Since $\text{rad}(A) = \text{rad}(A_1) \times \dots \times \text{rad}(A_t)$ we have $\mathfrak{m} \subseteq \text{rad}(A)$. Therefore H is distinguished.

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