A GENERALIZATION OF THE LITTLE THEOREM OF PICARD

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Dedicated to the Memory of C. D. Papakyriakopoulos

ABSTRACT. The method used by the author in cooperation with S. I. Goldberg and T. Ishihara in the study of harmonic mappings of bounded dilatation (J. Differential Geometry **10** (4) (1975)) is employed here to derive a Picard type theorem for quasiconformal mappings into spaces which are not necessarily hyperbolic. More precisely: if N is a nonpositively curved Riemannian manifold of dimension n and scalar curvature bounded away from zero, and if M is a complete locally flat Riemannian manifold of dimension $m \geq n$, then every harmonic K -quasiconformal mapping $f: M \rightarrow N$ is a constant mapping. If, in addition, M and N are Kaehler manifolds then every conformal mapping, $f: M \rightarrow N$ is of rank at most $n - 2$. For $m = n = 2$ we obtain the classical "little" theorem of Picard.

1. Introduction. The "little" theorem of Picard states that a holomorphic mapping of the complex plane to the complex plane with two lacunary values reduces to a constant function. We know that the complex plane $\mathbb{C}$ with two points deleted carries a Kaehler metric of constant negative curvature [7]. From this point of view, the following theorem extends Picard's theorem to mappings of bounded dilatation [3] (see also [4]).

THEOREM A. *Let M be a complete connected locally flat Riemannian manifold, and let N be a Riemannian manifold with negative sectional curvatures bounded away from zero. Then, if $f: M \rightarrow N$ is a harmonic mapping of bounded dilatation, it reduces to a constant mapping.*

In particular, this theorem holds for harmonic K -quasiconformal mappings.

S. Kobayashi, using the notion of hyperbolic manifolds, obtained the following Picard type theorem [7].

THEOREM B. *Let M' be a complex manifold on which a complex Lie group acts transitively, and let M be a hyperbolic manifold. Then, every holomorphic mapping $f: M' \rightarrow M$ is a constant mapping.*

Another extension of Picard's theorem, worded more closely to the classical

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one, was obtained by Fujimoto [2] and Green [5]:

THEOREM C. *Let $\mathbb{C}^m$ be the m -dimensional complex Euclidean space and let $\mathbb{P}_n \mathbb{C}$ be the n -dimensional complex projective space. Then every holomorphic mapping $f: \mathbb{C}^m \rightarrow \mathbb{P}_n \mathbb{C}$ that omits $2n + 1$ hyperplanes in general position must reduce to a constant mapping.*

We know that the n -dimensional complex projective space with $2n + 1$ hyperplanes in general position deleted is a hyperbolic manifold [6]. Thus, we see that in Theorems A, B, and C the target space is hyperbolic. It is in this sense that the following theorem gives a new extension of Picard's theorem.

THEOREM 1. *Let M be a complete locally flat Riemannian manifold of dimension m and let N be a nonpositively curved Riemannian manifold of dimension $n \leq m$. If the scalar curvature of N is bounded away from zero, then every harmonic K -quasiconformal mapping $f: M \rightarrow N$ is a constant mapping.*

Clearly, N need not be hyperbolic.

An immediate consequence of this theorem is

COROLLARY 1. *Let M be a complete locally flat Riemannian manifold of dimension m , and let N be a nonpositively curved Einstein manifold of dimension $2 < n \leq m$. If N is not locally flat, then every harmonic K -quasiconformal mapping $f: M \rightarrow N$ is a constant mapping.*

An example is given which shows that the restriction on the dimensions of M and N cannot be dropped.

2. K -quasiconformal harmonic mappings. We begin by reviewing some of the results obtained in [3], where detailed proofs can be found.

Let V and W be two Euclidean vector spaces of dimensions m and n , respectively. A linear mapping $F: V \rightarrow W$ of rank $k = \min(m, n)$ maps a $(k - 1)$ -dimensional sphere to a $(k - 1)$ -dimensional ellipsoid with semiaxes of lengths $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_k > 0$, where $\gamma_i^2 = \lambda_i$, $i = 1, 2, \dots, k$, are the eigenvalues of $'FF: V \rightarrow V$. The ratio $l = \gamma_1/\gamma_k$ will be called the *total dilatation* of F .

Let $f: M \rightarrow N$ be a C^∞ mapping and let $(f_\star)_x: T_x M \rightarrow T_{f(x)} N$ be the induced mapping of tangent spaces at x .

DEFINITION 1. *If either $(f_\star)_x = 0$ at each $x \in M$ or if the total dilatation $l(x)$ of $(f_\star)_x$ is bounded above by a constant K on M , then f is said to be a K -quasiconformal mapping [4].*

Using a moving orthonormal frame $\{e_1, e_2, \dots, e_m\}$ in M with coframe $\{\omega_1, \omega_2, \dots, \omega_m\}$ and corresponding Levi-Civita connection forms ω_{ij} , $i, j = 1, 2, \dots, m$, we have

$$ds^2 = \omega_1^2 + \omega_2^2 + \cdots + \omega_m^2,$$

and the structure equations

$$(1) \quad d\omega_i = \sum \omega_j \wedge \omega_{ji}, \quad d\omega_{ij} = \sum \omega_{ik} \wedge \omega_{kj} + \Omega_{ij},$$

where Ω_{ij} are the curvature forms. We write

$$\Omega_{ij} = -\frac{1}{2} \sum R_{ijkl} \omega_k \wedge \omega_l,$$

where R_{ijkl} are the components of the curvature tensor. The sectional curvature on the section defined by $\{e_i, e_j\}$ is given by $K_{ij} = R_{ijij}$ and the scalar curvature is defined by $R = \sum_{i,j} K_{ij} = \sum R_{ii}$, where R_{ij} is the Ricci curvature tensor defined by $R_{ij} = \sum R_{ikjk}$. We denote the corresponding elements of N with an asterisk. Thus, the Riemannian metric of N is given by $ds^{*2} = \sum \omega_a^{*2}$. (Here, and in the sequel, the indices run as follows: $i, j, k, \dots = 1, 2, \dots, m$ and $a, b, c, \dots = 1, 2, \dots, n$.) Under the mapping f ,

$$f_* e_i = \sum_a A_i^a e_a^*, \quad f^* \omega_a = \sum_i A_i^a \omega_i,$$

where f_* and f^* are the mappings induced by f on tangent vectors and forms.

The covariant differential of A_i^a is defined by

$$(2) \quad DA_i^a = dA_i^a + \sum A_j^a \omega_{ji} + \sum A_i^b \omega_{ba}^* \equiv \sum A_{ik}^a \omega_k \quad (\text{say}),$$

where $A_{ik}^a = A_{ki}^a$.

DEFINITION 2. The mapping f is called harmonic [1] if $\sum_i A_{ii}^a = 0$.

Differentiating (2) and using the structure equations (1) in M and N we obtain

$$(3) \quad \sum_j DA_{ij}^a \wedge \omega_j = \sum_j A_j^a \Omega_{ji} + \sum_b A_i^b \Omega_{ba}^*,$$

where

$$(4) \quad DA_{ij}^a \equiv dA_{ij}^a + \sum A_{kj}^a \omega_{ki} + \sum A_{ik}^a \omega_{kj} + \sum A_{ij}^b \omega_{ba}^* \equiv \sum A_{ijk}^a \omega_k \quad (\text{say}).$$

The Laplacian of the C^∞ function $\|f_*\|^2 = \sum_{a,i} (A_i^a)^2$ on M is given by

$$(5) \quad \Delta \|f_*\|^2 = \sum \nabla^2 \|f_*\| (e_k, e_k) = 2 \sum_{i,j,a} ((A_{ij}^a)^2 + A_i^a A_{jji}^a).$$

Using (2), (3), (4) and the structure equations, (5) becomes

$$(6) \quad \begin{aligned} \frac{1}{2} \Delta \|f_*\|^2 &= \sum_{a,i,j} (A_{ij}^a)^2 + \sum_{a,i,j} R_{ij} A_i^a A_j^a \\ &\quad - \sum_{a,b,c,d,i,j} R_{abcd}^* A_i^a A_j^b A_i^c A_j^d + \sum_{a,i,j} A_i^a A_{jji}^a, \end{aligned}$$

where the last term $\sum A_i^a A_{jji}^a$ vanishes if f is harmonic. If $\|f_*\|^2$ attains a maximum at $x \in M$, then from (6),

$$(7) \quad \sum_{a,i,j} R_{ij} A_i^a A_j^a - \sum_{a,b,c,d,i,j} R_{abcd}^* A_i^a A_j^b A_i^c A_j^d + \sum_{a,i,j} A_i^a A_{jji}^a \leq 0$$

at x .

Let B_α be the open ball in E^m of radius α and constant negative curvature $-4/\alpha^2$. Then, the metric

$$(8) \quad ds^2 = (\alpha^2/\alpha^2 - r^2)^2 \sum_i dx_i^2,$$

of B_α , $r^2 = x_1^2 + x_2^2 + \cdots + x_m^2$, is conformally related to the flat metric $d\bar{s}^2 = \sum_i dx_i^2$ of E^m . Let $f: B_\alpha \rightarrow N$ be a C^∞ mapping which is harmonic with respect to the metrics $(d\bar{s}^2, ds^{*2})$. Such a mapping is not, in general, harmonic with respect to the metrics (ds^2, ds^{*2}) , but the quantity $\sum A_i^a A_{jji}^a$ of (6) is never negative (see [3, Theorem 4.3]). If by $\|\cdot\|$ and $\|\cdot\|_\alpha$ we denote the norms with respect to the flat metric of B_α and its conformally related metric (8), respectively, we have

$$(9) \quad \|f_\star\|_\alpha^2 = (\alpha^2 - r^2/\alpha^2)^2 \|f_\star\|^2.$$

The function $\|f_\star\|_\alpha^2$ attains a maximum on the closure of B_α , but, as we see from (9), it becomes zero on the boundary. Hence, $\|f_\star\|_\alpha^2$ attains a maximum at $x(\alpha)$ in B_α unless it is identically zero. Thus, from (7),

$$(10) \quad - \sum_{a,b,c,d,i,j} R_{abcd}^* A_i^a A_j^b A_i^c A_j^d \leq - \sum_{a,i,j} R_{ij} A_i^a A_j^a$$

at $x(\alpha)$.

From (8), we have $R_{ikik} = -4/\alpha^2$. Hence, $R_{ij} = -4(m-1)/\alpha^2 \cdot \delta_{ij}$. Thus, from (10), the following proposition follows.

PROPOSITION 1. *The norm $\|f_\star\|_\alpha$ attains a maximum, denoted by $\|f_\star\|_{(\alpha)}$, at some point $x(\alpha) \in B_\alpha$, and at this point*

$$(11) \quad - \sum_{a,b,c,d,i,j} R_{abcd}^* A_i^a A_j^b A_i^c A_j^d \leq \frac{4(m-1)}{\alpha^2} \|f_\star\|_{(\alpha)}^2,$$

where R_{abcd}^* are the components of the curvature tensor of N at $f(x)$.

3. Picard's first theorem. Let $\mathbf{C}$ be the entire complex plane, and a, b two points of $\mathbf{C}$. Then, a holomorphic mapping $f: \mathbf{C} \rightarrow \mathbf{C} - \{a, b\}$ is a constant mapping. This is the classical first theorem of Picard. It is known that $\mathbf{C} - \{a, b\}$ can be given a Kaehler metric with curvature bounded above by -4 . In this sense, Theorem 1 gives an extension of Picard's theorem to more general mappings.

In order to prove this theorem we first establish some lemmas. The proof of the following lemma is elementary.

LEMMA 1. *Let V and W be Euclidean vector spaces, and let $F: V \rightarrow W$ be a*

linear mapping of rank ≥ 2 . If S is a two-dimensional subspace of $F(V)$, there is a pair of orthonormal vectors $\{X_1, X_2\}$ in V such that the pair $\{Y_1 = F(X_1), Y_2 = F(X_2)\}$ spans S and $Y_1 \perp Y_2$.

LEMMA 2. Let $f: M \rightarrow N$ be a K -quasiconformal mapping, and let X_1, X_2 be any two unit vectors of $T_x(M)$. If $A_1(x) = f_*(X_1) \neq 0$ and $A_2(x) = f_*(X_2) \neq 0$, then $1/K \leq \|A_1(x)\|/\|A_2(x)\| \leq K$ for every $x \in M$. Thus, if $\|A_1(x)\| \rightarrow 0$, then $\|A_2(x)\| \rightarrow 0$.

PROOF. If $l(x)$ is the total dilatation of $(f_*)_x$, and if $\|A_1(x)\| \neq 0$, $\|A_2(x)\| \neq 0$, then $1/l(x) \leq \|A_1(x)\|/\|A_2(x)\| \leq l(x)$. Since $l(x) \leq K$ the lemma follows.

LEMMA 3. Let $f: M \rightarrow N$ be a K -quasiconformal mapping, let $\{e_1, e_2, \dots, e_m\}$ be an orthonormal base of $T_x(M)$, and let $A_i = f_*(e_i)$, $i = 1, 2, \dots, m$. If $A_1 \perp A_2$, and if at least one of the $\|A_i\|$ is ≥ 1 , then

$$(12) \quad \|A_1 \wedge A_2\|^2 / \|f_*\|^2 \geq 1/(K^2 + (m-1)K^4).$$

PROOF. Since $A_1 \perp A_2$, we have $\|A_1 \wedge A_2\| = \|A_1\| \cdot \|A_2\|$. Hence,

$$(13) \quad \frac{\|A_1 \wedge A_2\|^2}{\|f_*\|^2} = \frac{\|A_2\|^2}{1 + \sum_{i=2}^m \|A_i\|^2 / \|A_1\|^2}.$$

If $\|A_j\|^2 \geq 1$, then from Lemma 2,

$$\|A_i\|^2 / \|A_1\|^2 \leq K^2, \quad \|A_2\|^2 \geq \|A_j\|^2 / K^2 \geq 1/K^2.$$

Substituting in (13), we obtain (12).

LEMMA 4. Let N be a nonpositively curved Riemannian manifold of dimension $n \geq 2$, and let E^m be m -dimensional Euclidean space, where $m \geq n$. If at each point of N there is at least one sectional curvature $\leq -A$, where A is a positive constant, then a K -quasiconformal harmonic mapping $f: E^m \rightarrow N$ is a constant mapping.

PROOF. Consider the mapping $f: B_\alpha \rightarrow N$. According to Proposition 1, $\|f_*\|_\alpha$ attains a maximum $\|f_*\|_{(\alpha)}$ at a point $x(\alpha) \in B_\alpha$. By hypothesis, there is a section $S_{f(x(\alpha))}N$ on which N has sectional curvature $\leq -A$, where A is a positive constant. If $\|f_*\|_{(\alpha)} \neq 0$, we have $T_{f(x(\alpha))}N = f_* T_{x(\alpha)}B_\alpha$, since f_* is of maximal rank. From Lemma 1, we can then select an orthonormal frame $\{X_1, X_2, \dots, X_m\}$ at $x(\alpha) \in B_\alpha$ such that the pair $A_1 = f_* X_1$, $A_2 = f_* X_2$ spans $S_{f(x(\alpha))}$ and $A_1 \perp A_2$. The sectional curvature of N along $S_{f(x(\alpha))}$ is given by

$$H_{1,2} = \sum_{a,b,c,d} R_{abcd}^* A_1^a A_2^b A_1^c A_2^d / \|A_1 \wedge A_2\|^2.$$

If $-b_{ij} \leq 0$ are the sectional curvatures along the sections spanned by A_i, A_j

for $i, j = 1, 2, \dots, m$, then $H_{1,2} \leq -A < 0$, and $-b_{ij} \leq 0$, $(i, j) \neq (1, 2)$. Hence,

$$(14) \quad \sum_{a,b,c,d,i,j} R_{abcd}^* A_i^a A_j^b A_i^c A_j^d \leq -2A \|A_1 \wedge A_2\|^2 - 2 \sum_{(i,j) \neq (1,2)} b_{ij} \|A_i \wedge A_j\|^2.$$

Inequality (11) of Proposition 1 together with (14) yields

$$\frac{2(m-1)}{\alpha^2} \|f_\star\|_{(\alpha)}^2 \geq A \|A_1 \wedge A_2\|^2 + \sum_{(i,j) \neq (1,2)} b_{ij} \|A_i \wedge A_j\|^2,$$

from which

$$(15) \quad (2(m-1)/\alpha^2) \|f_\star\|_{(\alpha)}^2 \geq A \|A_1 \wedge A_2\|^2 \quad \text{at } x(\alpha).$$

If $\|f_\star\|_{(\alpha)}$ is not bounded as $\alpha \rightarrow \infty$, then at least one of the A_i has norm ≥ 1 for large values of α . Hence, Lemma 3 and (15) yield

$$(16) \quad 2(m-1)/\alpha^2 \geq A/(K^2 + (m-1)K^4).$$

If we let $\alpha \rightarrow \infty$, then (16) implies $A = 0$, which contradicts the assumption on A . If $\|f_\star\|_{(\alpha)}$ is bounded for all α , then from (15) and Lemma 2, $\|A_i\| \rightarrow 0$ for all i at $x(\alpha)$ as $\alpha \rightarrow \infty$. Thus, $\|f_\star\|_\alpha \rightarrow 0$ at each point of B_α as $\alpha \rightarrow \infty$. But, from (9), $\lim_{\alpha \rightarrow \infty} \|f_\star\|_\alpha^2 = \|f_\star\|^2$ at each point $x \in B_\alpha$. Hence, $\|f_\star\| = 0$.

PROOF OF THEOREM 1. Let N be a nonpositively curved Riemannian manifold with scalar curvature R such that $R \leq -B$, where B is a positive constant. Then there is at least one sectional curvature $\leq -B/n(n-1)$ at each point of N . Let M be a complete connected locally flat Riemannian manifold. We know that the universal covering of M is E^m . Let $\tilde{f} = f \circ \pi: E^m \rightarrow N$, where $\pi: E^m \rightarrow M$, is the covering map. Then $\|\tilde{f}_\star(x)\| = \|f_\star(\pi(x))\|$. Thus, the theorem follows from Lemma 4.

PROOF OF COROLLARY 1. In an n -dimensional Einstein manifold, $n > 2$, the scalar curvature R is constant. Since N is not flat, $R < 0$, and the proof is a consequence of Theorem 1.

COROLLARY 2. Let M and N be Kaehlerian manifolds, which as Riemannian manifolds, are as in Theorem 1. Then every conformal mapping $f: M \rightarrow N$ is of rank at most $n-2$.

REMARK. We note here that Picard's classical theorem is contained in the statement of Corollary 2. For then $M = \mathbb{C}$, with the flat metric, $N = \mathbb{C} - \{a, b\}$ with a Kaehler metric of negative curvature bounded away from zero and $n-2 = 0$.

COROLLARY 3. Let M be a complete connected locally flat Kaehler manifold, and let $\mathbb{C}^n$ be the usual n -dimensional complex space. Then a conformal mapping

$f: M \rightarrow \mathbf{C}^n$ of rank n cannot miss two parallel hyperplanes.

PROOF. Let L_1 and L_2 be two parallel hyperplanes missed by f . By a linear change of coordinates in $\mathbf{C}^n$ the hyperplanes L_1 and L_2 may be assumed to be $z_1 = a$ and $z_1 = b$. Thus, $\mathbf{C}^n - L_1 \cup L_2$ is biholomorphic to $\mathbf{C}^{*n} = (\mathbf{C} - \{a, b\}) \times \mathbf{C}_2 \times \cdots \times \mathbf{C}_n$, where $\mathbf{C}_i = \mathbf{C}$, $i = 2, \dots, n$. We know that $\mathbf{C} - \{a, b\}$ can be given a Kaehler metric $ds_1^2 = 2g dz_1 d\bar{z}_1$ with Gaussian curvature bounded above by -4 . Hence, if the metric of $\mathbf{C}^{*n}$ is defined by

$$ds^2 = ds_1^2 + \sum_{i=2}^n dz_i d\bar{z}_i,$$

its scalar curvature is negative and bounded away from zero. Consequently, by Theorem 1, the mapping is constant.

REMARKS. 1. Let $N = \mathbf{C}^* \times \mathbf{C}$, where $\mathbf{C}^* = \mathbf{C} - \{0, 1\}$ with a Kaehler metric ds_1^2 with Gaussian curvature bounded above by -4 , and $\mathbf{C}$ is the complex plane with the flat metric $ds_2^2 = |dz_2|^2$. The metric of N is defined as the product metric $ds^2 = ds_1^2 \times ds_2^2$. Then, the scalar curvature of N is bounded above by -4 . If $\varphi(z)$ is a nonconstant holomorphic function, the mapping $f: \mathbf{C} \rightarrow N$ defined by $z_1 = \text{const}$, $z_2 = \varphi(z)$, where z_1 is a coordinate of $\mathbf{C}^*$, is a nonconstant holomorphic mapping. This example shows that the condition $m \geq n$ of Theorem 1 cannot be dropped.

2. The author would like to thank the referee for the following example. Let $N = \mathbf{C}$ with the conformal metric $ds^2 = (1 + |z|^2) dz d\bar{z}$, and $M = \mathbf{C}$ with the flat metric. The curvature of N is strictly negative everywhere, but not bounded away from zero. The identity mapping is trivially conformal, which shows that the restriction on the curvature of N in Theorem 1 is essential.

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