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A CURIOSITY CONCERNING THE DEGREES OF THE CHARACTERS OF A FINITE GROUP

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ABSTRACT. Let G be a finite group with irreducible characters $\{\dots, \chi, \dots\}$ and $K = \mathbf{Q}(\dots, \chi, \dots)$ the field generated over the rationals by their values. We will prove:

If $K = \mathbf{Q}$ (*or if* $[K:\mathbf{Q}]$ *is odd*) *then* $\prod_{\chi(1)\ odd} \chi(1)$ *is a perfect square.*

More generally,

THEOREM. (a)

$$(-1)^{(\Sigma\chi(1)-(m+1))/2} \prod_{\chi(1)\ odd} \chi(1)$$

is a square in K if $|G|$ is even.

(b) $(-1)^{(|G|-1)/2}|G|$ *is a square in K if $|G|$ is odd.*

[Recall $|G| = \Sigma\chi(1)^2$ so that $|G| \equiv \Sigma\chi(1) \bmod 2$, and so $\Sigma\chi(1) \equiv m+1 \bmod 2$ where m denotes the number of involutions in G.]

PROOF OF THEOREM. Let F be any splitting field for G such that char $F \nmid |G|$. Consider the F-vector space FG on which we have the nondegenerate symmetric bilinear form defined by $B(g, h) = \rho(gh)$ for $g, h \in G$, where ρ is the trace of the regular representation of G. So

$$B(g,h) = \begin{cases} 0, & g \neq h^{-1}, \\ |G|, & g = h^{-1}, \end{cases}$$

and with respect to this basis of group elements, B is the direct sum of $(|G| - (m+1))/2$-matrices$\begin{pmatrix} 0 & |G| \\ |G| & 0 \end{pmatrix}$ and $m+1$ (1×1)-matrices $(|G|)$. So the discriminant of B is $(-1)^{(|G|-(m+1))/2}|G|^{|G|}$. But, if we identify FG with the direct sum $\Sigma M_{\chi(1)}(F)$ of $(\chi(1) \times \chi(1))$ matrix algebras over F, and if e^{χ}_{ij} denote the matrix units, then

$$B\left(e^{\chi}_{ij}, e^{\lambda}_{kl}\right) = \begin{cases} \chi(1), & i = l, j = k, \chi = \lambda, \\ 0, & \text{otherwise.} \end{cases}$$

So with respect to this basis, B is the direct sum of $(\chi(1)^2 - \chi(1))/2$-matrices $\begin{pmatrix} 0 & \chi(1) \\ \chi(1) & 0 \end{pmatrix}$ and $\chi(1)$ (1×1)-matrices $(\chi(1))$, and so has discriminant

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$$\prod_{\chi}(-1)^{(\chi(1)^2-\chi(1))/2}\left(\chi(1)^2\right)^{(\chi(1)^2-\chi(1))/2}\chi(1)^{\chi(1)}$$
$$= (-1)^{(|G|-\Sigma\chi(1))/2}\prod\chi(1)^{(\chi(1)^2)}.$$

Hence

$$(-1)^{(|G|-(m+1))/2}|G|^{|G|} \quad \text{and} \quad (-1)^{(|G|-\Sigma\chi(1))/2}\prod\chi(1)^{(\chi(1)^2)}$$

differ by a square in F.

If $|G|$ is odd, then the nontrivial characters occur in conjugate pairs, so that $\Pi\chi(1)^{(\chi(1)^2)}$ is a square; moreover, since $\Sigma\chi(1) = 1 + 2\Sigma_{i=1}^{(c-1)/2}(2k_i + 1)$ where c denotes the number of conjugacy classes,

$$\sum \chi(1) \equiv c \bmod 4.$$

Also, (Burnside) $|G| \equiv c \bmod 16$, so we have that $(-1)^{(|G|-1)/2}|G|$ is a square in F.

If char $F = 0$, then the intersection of all splitting fields F is just K. [Given a simple algebra finite dimensional over a number field L, the Grunwald-Wang and Tchebotarev Density Theorems imply the existence of a prime p in L and maximal subfields F_1 and F_2 such that p splits completely in F_1 but is divisible by only one prime of F_2; hence $F_1 \cap F_2 = L$. An analogous result holds for a semisimple algebra all of whose simple components have the same center; in the case of a group algebra, this common center can be chosen to be K above], and so our Theorem follows.

REMARKS. (1) If all the characters of G are real, we must have

$$(-1)^{(\Sigma\chi(1)-(m+1))/2} = 1, \quad \text{i.e.,} \sum \chi(1) \equiv m + 1 \bmod 4.$$

(2) If F can be chosen to be real, we have that

$$\sum \frac{\chi(1)^2 - \chi(1)}{2} = \frac{|G| - (m+1)}{2} = \text{Witt index of } B,$$

and so $\Sigma\chi(1) = m + 1$.

Both of these facts follow also from the classical Frobenius-Schur count of involutions.

(3) If G is cyclic of odd order n, then Theorem (b) states that

$$(-1)^{(n-1)/4}\sqrt{n} \in \mathbf{Q}(\xi_n).$$

(4) Recall that for even $|G|$, there are an odd number of nonprincipal, nonconjugate characters having both odd degree and an odd number of conjugates [Proc. Amer. Math. Soc. **30** (1971), 247–248]. In other words, $\Pi_{\chi(1)\ \mathrm{odd}}\chi(1)$ is nontrivial, and is never square solely by virtue of conjugate characters.

(5) It would be of interest to find necessary and sufficient conditions on G to insure that every character has an odd number of conjugates (this implies

that the characters are real) i.e., that $[K : \mathbf{Q}]$ is odd. It would also be of interest to determine exactly the discriminant of K. Also, it is unknown which abelian extensions of $\mathbf{Q}$ appear as $K = \mathbf{Q}(\ldots, \chi, \ldots)$ for some finite group G.

(6) We thank F. Gross and G. Walls for simplifying our original statement for odd $|G|$.

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