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ON THE TRACE OF AN IDEMPOTENT IN A GROUP RING

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ABSTRACT. Let KG be the group ring of a polycyclic by finite group G over a field K of characteristic zero. It is proved that if $e = \sum e(g)g$ is a nontrivial idempotent in KG then its trace $e(1)$ is a rational number r/s, $(r,s) = 1$, with the property that for every prime divisor p of s, G has an element of order p. This result is used to prove that if R is a commutative ring of characteristic zero, without nontrivial idempotents and G is a polycyclic by finite group such that no group order $\neq 1$ is invertible in R, then RG has no nontrivial idempotents.

1. Let KG be the group ring of a group G over a field K. By the trace of an element $\alpha = \sum_g \alpha(g)g$ of KG is understood $\alpha(1)$, the coefficient of the identity in α. The following two statements regarding the trace of an idempotent in KG are well known.

THEOREM (ZALESSKIĬ [7]). *The trace of an idempotent in KG lies in the prime subfield of K.*

THEOREM (KAPLANSKY, SEE [4]). *If K is a field of characteristic zero, the trace of a nontrivial idempotent in KG lies strictly between* 0 *and* 1.

We expect that in the characteristic zero case one should be able to say more, namely the denominator of the trace of a nontrivial idempotent is a $|G|$-number, in the sense that for every prime p dividing this denominator, G has an element of order p. This statement is proved in Theorem 1 for polycyclic by finite groups. We apply this to prove that if R is a unital commutative ring of characteristic zero without nontrivial idempotents, with the property that no group element $\neq 1$ has order invertible in R and G is polycyclic by finite, then RG has no nontrivial idempotents. This is proved for supersolvable groups in [3] and [5].

2. Results.

THEOREM 1. *Let KG be the group ring of a polycyclic by finite group G over a field K of characteristic zero. Let* $e = \sum_g e(g)g$ *be a nontrivial idempotent. Write*

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$e(1) = r/s$ with $(r, s) = 1$. If a prime p divides s, then there exists $1 \neq g \in G$ of p-power order with $e(g) \neq 0$.

THEOREM 2. *Let RG be the group ring of a polycyclic by finite group G over a commutative unital ring R of characteristic zero. Suppose that R has no nontrivial idempotents and that no group element $\neq 1$ has order invertible in R. Then RG has no nontrivial idempotents.*

3. **Proof of Theorem** 1. Denoting conjugate elements g and h of G by $g \sim h$, write for $\alpha = \sum_g \alpha(g)g \in KG$, $\tilde{\alpha}(g) = \sum_{h \sim g} \alpha(h)$, the sum of coefficients of all conjugates of g in α. The following result is well known.

LEMMA 1 (FORMANEK [2]). *If $e = \sum e(g)g = e^2 \in KG$ and G is Noetherian then $\tilde{e}(g) = 0$ for g of infinite order.*

Now let us suppose that G/A is finite and A is polycyclic. Then the number of infinite cyclic factors in any normal series of A is invariant, called the Hirsch number of G. We shall prove Theorem 1 by induction on the Hirsch number of G. We shall prove that if p is a prime divisor of s then there is an element $g \in G$ of p-power order with $\tilde{e}(g) \neq 0$, $g \neq 1$.

Suppose that the Hirsch number of G is $\geqslant 1$; then it is easy to see by induction on the solvability length of A that G has a torsion-free normal subgroup $N \neq \{1\}$ and therefore G/N has smaller Hirsch number. Let $\bar{e}$ be the image of e under the natural map $KG \to K(G/N)$. Then due to Lemma 1, $\bar{e}(1) = e(1) = r/s$. Therefore, by induction, there is a $\bar{g} \in G/N$ of p-power order such that $0 \neq \tilde{\bar{e}}(\bar{g}) = \sum \tilde{e}(h)$, a sum over certain h such that $\bar{h}$ is conjugate to $\bar{g}$.

Since again due to Lemma 1, $\tilde{e}(h) = 0$ for elements h of infinite order, we have that $\tilde{e}(h_0) \neq 0$ for some h_0 of finite order. This h_0 clearly has p-power order. Thus it remains to prove

LEMMA 2. *If G is finite and p is a prime divisor of s then there exists a $1 \neq g \in G$ of p-power order such that $\tilde{e}(g) \neq 0$.*

We shall need

LEMMA 3 (HATTORI [1]). *Suppose that $e = \sum e(g)g = e^2 \in KG$ where G is finite and K has characteristic zero. Let χ be the character of G afforded by KGe. Then for $g \in G$ we have $|C_G(g)|\tilde{e}(g) = \chi(g^{-1})$ where $C_G(g)$ denotes the centralizer of g in G.*

PROOF. For any $\alpha \in KG$ and $h \in G$, let $T_\alpha(h)$: $KG \to KG$ be the K-linear map which sends y to $hy\alpha$. Then $T_e(h)$ acts on KGe as left multiplication by h and annihilates $KG(1-e)$. Since $KG = KGe \oplus KG(1-e)$, choosing a suitable basis of KG, we see that the trace of the linear transformation $T_e(h)$ is equal to $\chi(h)$.

Now $T_e(h) = \sum_g e(g)T_g(h)$ and $T_g(h)$ sends x to hxg for any $x \in G$. Therefore $T_g(h)$ permutes the elements of G, so its trace is the number of $x \in G$ with $x = hxg$. But $x = hxg$ if and only if $x^{-1}h^{-1}x = g$. So the trace of $T_g(h)$ is 0 if and only if g is not conjugate to h^{-1} and is $|C_G(h)|$ otherwise. Hence

$$\chi(h) = \sum_{g \sim h^{-1}} e(g)|C_G(h)| = \tilde{e}(h^{-1})|C_G(h)|.$$

PROOF OF LEMMA 2. Suppose on the contrary that $\tilde{e}(g) = 0$ for all p-elements $g \neq 1$. Let P be a Sylow p-subgroup of G and let χ_P be the restriction of χ to P. Then $\chi_P(g) = 0$ for all $1 \neq g \in P$ by Lemma 3. Therefore,

$$(\chi(1)/\zeta(1))\zeta(g) = \chi_P(g) \quad \text{for all } g \in P,$$

where ζ is the character of the regular representation of P. Since the 1-representation occurs once and only once as a component of the regular representation, it follows that $\chi(1)/\zeta(1)$ is an integer. Thus $\chi(1)$ is a multiple of $|P|$. But $\chi(1) = |G|r/s$, so p cannot divide s and the lemma is proved.

4. **Proof of Theorem** 2. We may assume (as in [6]) that R is a Noetherian ring without nilpotent elements and therefore is contained in a direct sum of fields; $R \subseteq F_1 \oplus \cdots \oplus F_k \oplus \cdots \oplus F_r$ where F_i are of characteristic zero for $1 \leqslant i \leqslant k$ and F_{k+j} are of finite characteristic. Let $\Pi_i\colon RG \to F_iG$ be the map induced from the natural projection $R \to F_i$. We claim that $e(1)$ is of the form

$$e(1) = (r/s, r/s, \ldots, r/s, \alpha_1, \alpha_2, \ldots)$$

where the first k components are equal to the rational number r/s. This is true for finite G (see [6]). For polycyclic by finite groups it follows by induction on the Hirsch number in view of Lemma 1. Also, by Zalesskiĭ's Theorem, α_i's belong to finite fields.

We may suppose by considering $1 - e$ if necessary that $r/s \neq 0$. Since r and s can be taken to be relatively prime, there exist integers a and b such that $ar + bs = 1$. Thus

$$\beta = ae(1) + b1_R = (1/s, \ldots, 1/s, a\alpha_1 + b, a\alpha_2 + b, \ldots) \in R.$$

We may suppose that $a\alpha_i + b \neq 0$ for any i, as otherwise a suitable power of $s\beta$ is a nontrivial idempotent in R. Now,

$$s\beta - 1 = (0, 0, \ldots, 0, s(a\alpha_1 + b) - 1, s(a\alpha_2 + b) - 1, \ldots) \in R.$$

Again, by the same argument, $s(a\alpha_i + b) - 1 = 0$ for all i and so $a\alpha_i + b = 1/s$. We have

$$\beta = (1/s, \ldots, 1/s) = 1/s \cdot 1_R \in R.$$

Since s is a $|G|$-number as seen by applying Theorem 1 to $\Pi_1(e)$, it follows that $s = 1$. Therefore, $e = (1, 1, \ldots, 1, \alpha_1, \alpha_2, \ldots)$. Write $e' = 1 - e$. Then since $\Pi_i(e')$ has trace 0, it follows that $\Pi_i(e') = 0$ for $1 \leqslant i \leqslant k$. Hence $e' \in SG$ where $S = F_{k+1} \oplus \cdots \oplus F_r$. Let I be the ideal of R generated by the coefficients of e'. Then $I^2 = I \subset S$. By Krull's theorem, there exists an element $\gamma \in I$ such that $I(1-\gamma) = 0$. Thus $\gamma^2 = \gamma \in R$ and so $\gamma = 0$ or 1. Since clearly $\gamma \neq 1$ as $I \subset S$, we have $\gamma = 0$ and hence $I = 0$. It follows that $e' = 0$ and $e = 1$.

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