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A COUNTABLY COMPACT k'-SPACE NEED NOT BE COUNTABLY BI-k

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ABSTRACT. An example is given of a countably compact k'-space that is not countably bi-k. Interest for this example arises from a recent paper of Michael, Olson, and Siwiec and from a 1972 paper of E. Michael, both of which discuss mapping characterizations of a range space. The construction of the example assumes the continuum hypothesis.

1. **Introduction.** In recent years Arhangel'skiĭ [1], [2], Siwiec [10], and Michael [6] characterized images of certain kinds of spaces under certain kinds of mappings. These characterizations are summarized in Table 1 of [6], part of which we now reproduce:

	B	C	D	E
(4)	strongly k'	countable base	countably bi-sequential	countably bi-k
(5)	k'	Fréchet $\aleph_0$	Fréchet	singly bi-k

All the terms in this table are defined in [6]. Those terms having a direct bearing on the example in this paper will be defined in §2. The two entries in each of Columns C and D coincide in the presence of countable compactness; for Column C this was proved in [5, (D) and (C), p. 983], and for Column D in [3, Corollary 3 to Theorem 7] and in [8, Theorem 5.1]. In this note, we show this is not the case in Columns B and E. (It seems rather plausible that this should be so, because [7, Diagrams 1.2 and 1.3, Proposition 2.4] imply that, under rather mild restrictions (e.g., X Lindelöf, or Y a Fréchet space, or every $y \in Y$ a G_δ), Y countably compact and regular implies that every hereditarily quotient map $f: X \to Y$ is countably bi-quotient.[1] Nevertheless, the answer for both Columns B and E is, in general, negative, as the following example shows. The construction of the example assumes the continuum hypothesis, which we indicate by [CH].

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[1] The entries in Row 4 (respectively, Row 5) of [6, Table 1] are the images under countably bi-quotient (respectively, hereditarily quotient) maps of certain kinds of spaces. Those spaces are: (B) locally compact, paracompact spaces, (C) separable metrizable spaces, (D) metrizable spaces, and (E) paracompact M-spaces.

EXAMPLE 1.1 [CH]. There exists a completely regular, countably compact k'-space Y that is not countably bi-k.

As the proof will show, Y actually has the following property, which is stronger than countable compactness: Every infinite subset of Y has an infinite subset with compact closure.

2. **Some preliminaries.** We begin with the appropriate definitions. A space Y is a *k'-space* [2, Chapter III, Definition 3.2] if whenever $A \subset Y$ and $y \in \bar{A}$ (the closure of A), then there exists a compact set $K \subset Y$ such that $y \in (A \cap K)^{-}$. A space Y is *countably bi-k* [6, Definition 4.E.1] if whenever (A_n) is a decreasing sequence of subsets with a common accumulation point y, then there exists a k-sequence (K_n) such that $y \in (A_n \cap K_n)^{-}$ for every n. Here (K_n) is called a *k-sequence* if: (1) $K_n \supset K_{n+1}$ for every n, (2) $K = \bigcap_n K_n$ is compact, and (3) if $U \supset K$ and U is open, then $U \supset K_n$ for some n.

In the construction of Example 1.1, the Stone-Čech compactification βN of the set N of nonnegative integers (with the discrete topology) plays a central role, and we make the convention that, for subsets of βN, "closure" means closure in βN. Thus the "closure" of any such set is necessarily compact. Following [4, 6.S, pp. 98–99], A' shall denote the boundary in βN of a subset A of N. We remark that $N' = \beta N \backslash N$ is closed in βN and refer to [4, ibid.] for more details.

It is worthwhile to establish some lemmas to aid in the construction of the example, the first of these lemmas to be of a set-theoretic nature.

LEMMA 2.1. *Let $\{W_\alpha\}_{\alpha<\Omega}$ be a strictly decreasing family of sets indexed by the first uncountable ordinal Ω, let W denote $\bigcap_\alpha W_\alpha$, and let $\{G_\alpha\}_{\alpha<\Omega}$ be any family of sets satisfying $G_\alpha \cap W_\beta \backslash W \neq \varnothing$ for all α and β. Then there exists an increasing function $\phi\colon [0,\Omega) \to [0,\Omega)$ such that $G_\alpha \cap W_{\phi(\alpha)} \backslash W_{\phi(\alpha+1)} \neq \varnothing$ for all α.*

PROOF. Let $\phi(0) = 0$ and $x_0 \in G_0 \cap W_{\phi(0)} \backslash W$. Suppose inductively that $x_\alpha \in G_\alpha \cap W_{\phi(\alpha)} \backslash W$ for every $\alpha < \beta < \Omega$, and whenever $\alpha < \gamma < \beta$, we have $\phi(\alpha) < \phi(\gamma)$ and $x_\alpha \notin W_{\phi(\gamma)}$. Define

$$\phi(\beta) = \sup\{\min\{\gamma\colon x_\alpha \notin W_\gamma\}\colon \alpha < \beta\}$$

and let $x_\beta \in G_\beta \cap W_{\phi(\beta)} \backslash W$. Then ϕ is as required.

LEMMA 2.2. *If A and B are F_σ subsets of βN, and if $\bar{A} \cap B = A \cap \bar{B} = \varnothing$, then $\bar{A} \cap \bar{B} = \varnothing$.*

PROOF. Since βN is normal, an easy induction shows that A and B can be separated by disjoint open sets U and V. Since βN is extremally disconnected (see [4, 6.M, p. 96]), $\bar{U} \cap \bar{V} = \varnothing$.

We complete the preliminaries with the following technical lemma.

LEMMA 2.3 [CH]. *Suppose S is an open F_σ in N'. Then there exist a decreasing family $\{V_\alpha\}_{\alpha<\Omega}$ of open-closed subsets of N', and a subset P of $N' \backslash \bar{S}$ satisfying the following conditions*:

(a) $\{V_\alpha\}_{\alpha<\Omega}$ *is a base for the neighborhoods of* $\overline{S}$ *in* N'.

(b) $P\backslash V_\alpha$ *is countable for all* α.

(c) P *is relatively discrete and, hence, nowhere dense in* N'.

(d) *If* $A \subset N'$, *if* $A\backslash V_\alpha$ *is countable for all* α, *and if* $A \cap \overline{P} \subset \overline{S}$, *then* $\overline{A} \cap \overline{P} \subset \overline{S}$.

(e) *If* F *is open-closed in* N', *and if* $(F\backslash\overline{S})^- \cap \overline{S} \neq \varnothing$, *then* $F \cap P \neq \varnothing$.

PROOF. Let $\mathcal{U}$ be the collection of all open-closed sets in N' that contain S. Write $\mathcal{U} = \{U_\alpha: \alpha < \Omega\}$, and let $B_\alpha = N'\backslash U_\alpha$. For each $\beta < \Omega$, S and $\bigcup\{B_\alpha: \alpha < \beta\}$ are disjoint open F_σ's and hence have disjoint closures. Since these closures are compact, there exists an open-closed set in N' containing S and disjoint from $\bigcup\{B_\alpha: \alpha < \beta\}$. That is, $\bigcap\{U_\alpha: \alpha < \beta\}$ contains some U_γ. Let $W_0 = U_0$ and let W_β be the first U_γ contained in $U_\beta \cap (\bigcap\{W_\alpha: \alpha < \beta\})$. Then $\{W_\alpha\}_{\alpha<\Omega}$ is a decreasing family of open-closed subsets of N' and a base for the neighborhoods of $\overline{S} = \bigcap\{W_\alpha: \alpha < \Omega\}$, since each W_α is compact.

Let $\mathcal{F}$ be the collection of all open-closed sets F in N' such that $(F\backslash\overline{S})^- \cap \overline{S} \neq \varnothing$. Write $\mathcal{F} = \{F_\alpha: \alpha < \Omega\}$. Since $F_\alpha\backslash\overline{S}$ is open in N', the set G_α of P-points in $F_\alpha\backslash\overline{S}$ is dense in $F_\alpha\backslash\overline{S}$ (see [4, 6.V, p. 100]). Thus $\overline{G}_\alpha \cap \overline{S} \neq \varnothing$ and hence $G_\alpha \cap W_\beta \neq \varnothing$ for all α and β. By Lemma 2.1, there exists an increasing function ϕ: $[0,\Omega) \to [0,\Omega)$ such that for all α, $G_\alpha \cap W_{\phi(\alpha)}\backslash W_{\phi(\alpha+1)} \neq \varnothing$. Let $V_\alpha = W_{\phi(\alpha)}$, and let P be the set obtained by choosing one P-point from each $F_\alpha \cap V_\alpha\backslash V_{\alpha+1}$. Thus (a), (b), and (e) are clearly satisfied. Since the sets $V_\alpha\backslash V_{\alpha+1}$ are open and pairwise disjoint, P is relatively discrete.

Since P is relatively discrete, P is locally compact and, hence, open in $\overline{P}$. If the interior in N' of $\overline{P}$ were nonempty, then its intersection with P would be a nonempty open subset of N', in which case N' would have isolated points (again because P is relatively discrete), which it does not. Thus P is nowhere dense, and (c) is satisfied.

For (d), suppose $A \subset N'$, $A\backslash V_\alpha$ is countable for all α, and $A \cap \overline{P} \subset \overline{S}$. If $x \in (\overline{A} \cap \overline{P})\backslash\overline{S}$, then $x \notin V_\alpha$ for some $\alpha < \Omega$. Then x belongs to the closures of the two countable sets $A\backslash V_\alpha$ and $P\backslash V_\alpha$. Since a P-point cannot be an accumulation point of a countable set in N', $(A\backslash V_\alpha)^- \cap P = \varnothing$. Since $A \cap \overline{P} \subset \overline{S}$, $\overline{P} \cap A\backslash V_\alpha = \varnothing$. By Lemma 2.2, $(A\backslash V_\alpha)^- \cap (P\backslash V_\alpha)^- = \varnothing$, contradicting the fact that x is in this intersection. Hence $(\overline{A} \cap \overline{P})\backslash\overline{S} = \varnothing$, completing the proof of the lemma.

3. **The construction and proof of the example.** Partition N into an infinite collection of infinite subsets: $N = \bigcup_{n=1}^\infty S_n$, each S_n infinite, and $S_n \cap S_m = \varnothing$ if $n \neq m$. Let $S = \bigcup_{n=1}^\infty S'_n$. Then S is an open F_σ in N'. Assuming [CH], let $\{V_\alpha\}_{\alpha<\Omega}$ and P be as in Lemma 2.3.

Define $X = \beta N\backslash(\overline{P}\backslash\overline{S})$, define $Y = X/\overline{S}$ ($\overline{S}$ is identified to a point), and let f denote the (necessarily perfect) identification map f: $X \to Y$. Let s denote $f(\overline{S})$, considered as an element of Y. Observe $Y\backslash\{s\}$ is homeomorphic to $\beta N\backslash(\overline{P} \cup \overline{S})$, and hence every point of $Y\backslash\{s\}$ has an open-closed, compact neighborhood.

First, we show X, and hence Y, is countably compact. Suppose $A \subset X$ is countably infinite. We shall produce an infinite subset B of A with $\overline{B}$ (necessarily compact) a subset of X.

Case 1. $A \cap N$ is infinite: Since $(A \cap N)'$ is open in N' and P is nowhere dense in N', $(A \cap N)' \setminus \overline{P}$ is nonempty and open in N'. Hence there is an infinite subset B of $A \cap N$ such that $\overline{B} \cap \overline{P} = \varnothing$. Thus $\overline{B}$ is a compact subset of X.

Case 2. $A \cap N'$ is infinite: In this case, let $B = A \cap N'$. Then $\overline{B} \subset X$ by Lemma 2.3(d).

Next, Y is a k'-space. Let $A \subset Y$ and y an accumulation point of A. We must produce a compact set $K \subset Y$ such that y is an accumulation point of $A \cap K$. If $y \neq s$, then y has a compact neighborhood and there is nothing to prove. We assume $y = s$ and $y \notin A$. Then $f^{-1}(A)^{-} \cap \overline{S} \neq \varnothing$.

Case 1'. $(f^{-1}(A) \cap N)^{-} \cap \overline{S} \neq \varnothing$: Then $(f^{-1}(A) \cap N)'$ intersects S_n' for some n, so $f^{-1}(A) \cap S_n$ is infinite for this n, in which case $K = f(\overline{S}_n)$ is compact and $y \in (A \cap K)^{-}$.

Case 2'. $(f^{-1}(A) \cap N')^{-} \cap \overline{S} \neq \varnothing$: Let $x_\alpha \in f^{-1}(A) \cap V_\alpha$ for each $\alpha < \Omega$. Then $\{x_\alpha : \alpha < \Omega\} \setminus V_\beta$ is countable for all β, and $\{x_\alpha : \alpha < \Omega\} \cap \overline{P} = \varnothing$, so by Lemma 2.3(d), $\{x_\alpha : \alpha < \Omega\}^{-} \subset X$. Then $K = f(\{x_\alpha : \alpha < \Omega\}^{-})$ is as required.

Finally, Y is not countably bi-k. Let $A_n = \bigcup_{k \geqslant n} f(S_k)$. Then $s \in \overline{A}_n$ for all n. Suppose (K_n) is a k-sequence with $s \in \bigcap_n K_n = K$, where K is compact and $s \in (A_n \cap K_n)^{-}$ for all n. Let $B_n = f^{-1}(A_n \cap K_n)$. Then $B_n' \cap \overline{S} \neq \varnothing$ for all n.

We claim $B_n' \cap P \neq \varnothing$ for all n. By Lemma 2.3(e), it suffices to show $(B_n' \setminus \overline{S})^{-} \cap \overline{S} \neq \varnothing$ for each n. If this intersection were empty for some n, then for some α, $(B_n' \setminus \overline{S}) \cap V_\alpha = \varnothing$, so $B_n' \cap V_\alpha$ is a compact subset of $\overline{S}$. Since $B_n' \cap V_\alpha$ is open in N', and $N' \setminus S$ is a G_δ, $B_n' \cap V_\alpha \cap N' \setminus S$ is a G_δ in N' that is contained in $\overline{S} \setminus S$, which has void interior in N'. Therefore $B_n' \cap V_\alpha \cap N' \setminus S = \varnothing$. That is, $B_n' \cap V_\alpha \subset S = \bigcup_k S_k'$, and since we have an open (in N') cover of a compact set, $B_n' \cap V_\alpha \subset \bigcup_{k<m} S_k'$ for some m. Then

$$B_{n+m}' \cap V_\alpha \subset \bigcup_{k<m} S_k' \cap (f^{-1}(A_{n+m}))' = \varnothing,$$

from which it follows that $B_{n+m}' \cap \overline{S} = \varnothing$, a contradiction.

Let Q be the set obtained by choosing one point from each $B_n' \cap P$. Then $\overline{Q} \subset \overline{P} \setminus \overline{S}$ by Lemma 2.2. Hence, $\overline{Q} \cap X = \varnothing$, so $\overline{Q} \cap f^{-1}(K) = \varnothing$. Separating these two compact subsets of βN, there exists an open-closed set $U \subset \beta N$ such that $\overline{Q} \subset U$ and $f^{-1}(K) \cap U = \varnothing$. Then $f(U \cap X)$ is closed in Y, disjoint from K, but intersects every K_n, since $Q \subset U$ implies $U \cap B_n \neq \varnothing$ and, hence, $f(U \cap X) \cap (A_n \cap K_n) \neq \varnothing$. That contradicts (K_n) being a k-sequence and completes the proof.

4. **Concluding comment.** We conclude with a comment about the other entries in Column E of [6, Table 1]. Those other entries are: (1) paracompact

M-spaces, (2) spaces of pointwise countable type, (3) bi-k-spaces, and (6) k-spaces. We do not know whether there is an example of a countably compact bi-k-space that is not of pointwise countable type. Otherwise, none of the entries in Column E of [6, Table 1] coincide in the presence of countable compactness. If Y is the space of Example 1.1, then $Y \times I$ is a countably compact k-space that is not singly bi-k (using [6, Proposition 4.E.4]). Arhangel'skiĭ [3, p. 1187] has an interesting example of a sequentially compact countably bi-k-space that is not bi-k, and the ordinal space $[0, \Omega)$ is a countably compact space of pointwise countable type that is not a paracompact M-space. Other examples regarding [6, Table 1] are mentioned in [6] and [8].

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