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DERIVATIONS, HOMOMORPHISMS, AND OPERATOR IDEALS

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ABSTRACT. Let $\mathfrak{A}$ be a C^*-algebra of operators on a Hilbert space, and let C_p be the Schatten p-ideal. It is shown that every derivation from $\mathfrak{A}$ to C_p is inner. A similar argument shows that two C^*-homomorphisms which agree modulo C_p are equivalent.

It is well known [7] that every derivation on a von Neumann algebra is inner. In addition, if $\mathfrak{A}$ is a C^*-algebra of operators and D is a derivation on $\mathfrak{A}$, then D extends to the von Neumann algebra generated by $\mathfrak{A}$ and so D is "almost inner." Here the term *derivation* refers to a linear transformation D from $\mathfrak{A}$ to $\mathfrak{A}$ satisfying $D(AB) = AD(B) + D(A)B$ for each A and B in $\mathfrak{A}$, and D is *inner* provided there is a T in $\mathfrak{A}$ satisfying $D(A) = AT - TA = D_T(A)$. One consequence of these results says that if a *-automorphism ϕ of a von Neumann algebra $\mathfrak{A}$ has a derivation as a logarithm, then ϕ is inner in the sense that there is a unitary operator U in $\mathfrak{A}$ satisfying $\phi(A) = U^*AU$ for each A in $\mathfrak{A}$.

The derivation equation makes sense for linear maps D from the C^*-algebra $\mathfrak{A}$ to a two sided $\mathfrak{A}$-module $\mathfrak{I}$. Here again it can be asked if such derivations are inner; that is, are they induced by an element of $\mathfrak{I}$ as above? In fancier language, the question asks if the cohomology group $H^1(\mathfrak{A}, \mathfrak{I})$ is trivial [2]. In this paper we show that D is inner provided $\mathfrak{A}$ is a C^*-subalgebra of the algebra $L(H)$ of all operators on a separable Hilbert space H, and $\mathfrak{I}$ is the Schatten p ideal C_p, $1 \leqslant p < \infty$.

In contrast with the situation for derivations from an algebra to itself, our theorem does not directly give information about C^*-homomorphisms, but our technique of proof applies equally well to the study of homomorphisms. We show that if ϕ and ψ are representations of a C^*-algebra and if $\phi(A) - \psi(A)$ is in C_p with $\|\phi(A) - \psi(A)\|_p \leqslant \alpha\|A\| < \|A\|$ for each nonzero A in the algebra, then there is a unitary operator U with $1 - U$ in C_p and $\psi(A) = U^*\phi(A)U$. The theorem remains true, except for some finite-dimensional summands, if the norm condition is omitted.

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In the last section we discuss what happens in case $\mathcal{J}$ is all of $L(H)$ or the ideal K of compact operators.

I. Any derivation on a C^*-algebra $\mathfrak{A}$ can be extended to the C^*-algebra obtained by adjoining an identity to $\mathfrak{A}$ by defining $D(1) = 0$. Similarly a C^*-homomorphism ϕ on $\mathfrak{A}$ can be extended by defining $\phi(1) = 1$. Therefore we consider only C^*-algebras which contain the identity, and if the C^*-algebra is a subalgebra of $L(H)$, we assume that the identity is the identity operator on H. With this in mind, we remark that every C^*-algebra (with identity) is generated by its group $\mathfrak{U}$ of unitary elements.

In this section we deal with the von Neumann Schatten p-classes C_p, $1 \leqslant p < \infty$. We remark here that if $1 \leqslant p < p'$, then $C_p \subset C_{p'}$ and $\|T\|_{p'} \leqslant \|T_p\|$ for each T in C_p. ($\|\cdot\|_p$ denotes the norm on C_p.) The reader is referred to [1] for a discussion of these ideals. The primary tool for our first theorem is the Ryll-Nardzewski fixed point theorem [6]. This theorem states that if Q is a nonempty weakly compact convex subset of a locally convex Hausdorff linear topological space, and if G is a semigroup of weakly continuous affine maps on Q which is noncontracting, then there is a common fixed point for the maps in G. Here noncontracting means that for a, b in Q, $a \neq b$, there is a continuous seminorm ρ such that $\inf\{\rho(T(a) - T(b)) \colon T \in G\} > 0$. Application of the Ryll-Nardzewski theorem to derivation problems is suggested in [3].

THEOREM 1. *If $\mathfrak{A}$ is a C^*-subalgebra of $L(H)$ which contains the identity operator, and if D is a derivation from $\mathfrak{A}$ to C_p, $1 \leqslant p < \infty$, then D is inner. That is, there is a T in C_p such that $D = D_T$ and $\|T\|_p$ is less than or equal $\|D\|_p$, the norm of D as a linear transformation from $\mathfrak{A}$ to C_p.*

PROOF. The operator D is continuous as a map from $\mathfrak{A}$ to $L(H)$ [3] and so it is closed as a map from $\mathfrak{A}$ to C_p. The continuity of D follows from the closed graph theorem.

First consider the case $p > 1$, so that C_p is a reflexive Banach space with dual space C_q, $1/p + 1/q = 1$. Let $\mathfrak{U}$ be the unitary group of $\mathfrak{A}$, $K = \{U^* D(U) \colon U \in \mathfrak{U}\}$, and Q the closed convex hull of K in C_p. The set Q is bounded by $\|D\|_p$ and so, by the reflexivity of C_p, Q is weakly compact. For each U in $\mathfrak{U}$, define an affine map T_U on Q by $T_U(C) = U^*CU + U^*D(U)$. Then

$$\begin{aligned} T_U(V^*D(V)) &= U^*V^*D(V)U + U^*D(U) \\ &= U^*V^*(D(V)U + VD(U)) = U^*V^*D(VU). \end{aligned}$$

So T_U maps K to K and therefore Q onto Q. Furthermore,

$$\begin{aligned} T_U T_V(C) &= U^*[V^*CV + V^*D(V)]U + U^*D(U) \\ &= U^*V^*CVU + U^*V^*D(VU) = T_{UV}(C), \end{aligned}$$

so that $\{T_U \colon U \in \mathfrak{U}\}$ is a group. Clearly, the maps T_U are weakly continuous and if a and b are in Q,

$$\|T_U(a) - T_U(b)\|_p = \|U^*(a-b)U\|_p = \|a-b\|_p$$

so that the group is noncontracting. Hence, by the Ryll-Nardzewski fixed point theorem, there is a common fixed point T for the T_U. That is,

$$T = T_U(T) = U^*TU + U^*D(U)$$

or $D(U) = UT - TU$ for each U in $\mathfrak{U}$. But $\mathfrak{U}$ generates $\mathfrak{A}$, so $D = D_T$, and since T is in Q, $\|T\|_p \leqslant \|D\|_p$.

In case $p = 1$, then since $C_1 \subset C_q$ for $q > 1$, there is a T_q in C_q such that $D(A) = AT_q - T_qA$ for each A in $\mathfrak{A}$. Furthermore, if $q' > q$, $\|T_q\|_{q'} \leqslant \|T_q\|_q \leqslant \|D\|_q \leqslant \|D\|_1$. For each n, there is a sequence $\{T_{q_{n,m}}: m = 1,2,\ldots\}$ with $q_{n,m} > q_{n,m+1}$, which converges to an operator S_n in the weak* topology of $C_{1+1/n}$. Furthermore, the sequence $\{T_{q_{n+1,m}}\}$ can be chosen to be a subsequence of $\{T_{q_{n,m}}\}$. Note that $\|S_n\|_{1+1/n} \leqslant \|D\|_1$. But all of the S_n are the same, for if F is any finite rank operator,

$$\operatorname{tr}(S_nF) = \lim_{m\to\infty} \operatorname{tr}(T_{q_{n,m}}F) = \lim_{m\to\infty} \operatorname{tr}(T_{q_{n+1,m}}F) = \operatorname{tr}(S_{n+1}F).$$

(Here "tr" stands for the trace on C_1.) Call the common value T. Clearly $D = D_T$, $T \in \bigcap_{q>1} C_q$ and $\|T\|_q \leqslant \|D\|_1$ for $q > 1$. Writing $T = UP$ in its polar decomposition, then P is in C_q for each $q > 1$ and $\|P\|_q \leqslant \|D\|_1$. It follows that $P^q \in C_1$ and $\|P^q\|_1 = \|P\|_q^q \leqslant \|D\|_1^q$. From here, it is easy to verify that P^q converges to P in the weak* topology on C_1 as q decreases to 1. Consequently P, and therefore T, is in C_1 and $\|P\|_1 = \|T\|_1 \leqslant \|D\|_1$.

We remark that our proof for the case $p > 1$ applies to any continuous derivation of $\mathfrak{A}$ into a Banach $\mathfrak{A}$-module which is a reflexive Banach space. Proposition 3.7 of [2] also gives this result.

COROLLARY 2. *If $\mathfrak{A}$ is a C^*-subalgebra of $L(H)$ and if D is a derivation from $\mathfrak{A}$ to C_1, then* $\operatorname{tr}(D(A)) = 0$ *for each A in $\mathfrak{A}$.*

COROLLARY 3. *If $\mathfrak{A}$ is a C^*-subalgebra of $L(H)$, and if B is an operator which commutes with $\mathfrak{A}$ modulo C_p, $1 \leqslant p < \infty$, that is, if $AB - BA$ is in C_p for each A in $\mathfrak{A}$, then $B = A' + C$ where A' commutes with $\mathfrak{A}$ and C is in C_p.*

PROOF. The operator B defines a derivation D_B from $\mathfrak{A}$ to C_p. There is a C in C_p satisfying $D_B = D_C$, so $A' = B - C$ commutes with $\mathfrak{A}$.

COROLLARY 4. *An operator B commutes with a C^*-algebra $\mathfrak{A}$ modulo C_p if and only if it commutes with the weak closure of $\mathfrak{A}$ modulo C_p.*

II. Let $\mathfrak{A}$ be any C^*-algebra and ϕ a representation of $\mathfrak{A}$ on some Hilbert space H. If U is a unitary operator on H for which $1 - U$ is in C_p, then the representation ψ defined by $\psi(A) = U^*\phi(A)U$ for each A in $\mathfrak{A}$ is such that $\psi - \phi$ is in C_p, that is, $\psi(A) - \phi(A)$ is in C_p for each A in $\mathfrak{A}$. Alternatively, let ϕ_1 and ϕ_2 be two n-dimensional representations; then $\phi \oplus \phi_1$ and $\phi \oplus \phi_2$ are again representations which agree modulo C_p. We now show that these are the only two ways in which this can happen.

THEOREM 5. *Let $\mathfrak{A}$ be a C^*-algebra with identity and suppose ϕ and ψ are representations of $\mathfrak{A}$ on H such that $\phi - \psi$ is in C_p for some p, $1 \leqslant p < \infty$. Then there is a partial isometry W for which $1 - W$ is in C_p and $\phi(A)W = W\psi(A)$ for each A in $\mathfrak{A}$. Furthermore, the initial space M of W reduces $\psi(\mathfrak{A})$ and the final space N reduces $\phi(\mathfrak{A})$, $\phi|_N$ is equivalent to $\psi|_M$, and M and N have the same finite codimension.*

PROOF. As with Theorem 1, we first assume that $p > 1$. Let $K = \{1 - \phi(U^*)\psi(U)$: U unitary in $\mathfrak{A}\} \subset C_p$ and let Q be the closed convex hull of K. Since C_p is a reflexive Banach space, Q is compact in the weak topology on C_p. For C in Q and U in the unitary group $\mathcal{U}$ of $\mathfrak{A}$, define $T_U(C) = 1 - \phi(U^*)(1 - C)\psi(U)$. Then T_U is a weakly continuous affine map of Q to Q, $T_U T_V = T_{UV}$, and this action of the group $\mathcal{U}$ on Q is noncontracting. Therefore the Ryll-Nardzewski fixed point theorem applies, so there is a C in Q such that $T_U(C) = C$ for each U in $\mathcal{U}$. That is,

$$C = 1 - \phi(U^*)(1 - C)\psi(U) \quad \text{or} \quad T = \phi(U^*)T\psi(U)$$

where $T = 1 - C$. But $\mathcal{U}$ generates $\mathfrak{A}$, so $\phi(A)T = T\psi(A)$ for each A in $\mathfrak{A}$.

Writing $T = WP$ according to its polar decomposition, we have $P^2 = T^*T = 1 - C^* - C + C^*C$ or

$$1 - P^2 = (1 - P)(1 + P) = C^* + C - C^*C \quad \text{is in } C_p$$

and

$$1 - P = (1 + P)^{-1}(C^* + C - C^*C) \quad \text{is in } C_p.$$

Therefore

$$1 - W = (1 - P) - (1 - T^*)W \quad \text{is in } C_p.$$

That W has the remaining desired properties follows by standard arguments.

The case $p = 1$ is proved by a weak* approximation argument using operators $C_p = 1 - T_p$ much as was done for Theorem 1.

In some cases, Theorem 5 gives unitary equivalence. Preserving the hypothesis and the notation of that theorem, we have:

COROLLARY 6. *If in addition $\mathfrak{A}$ has no nonzero finite dimensional representations, then W is unitary.*

PROOF. The representations $\phi|_{N^\perp}$ and $\psi|_{M^\perp}$ are finite dimensional and so must be zero.

COROLLARY 7. *If $\|\phi(U) - \psi(U)\|_p \leqslant \alpha < 1$ for each U in $\mathcal{U}$, then W is unitary.*

PROOF. From $\|\phi(U) - \psi(U)\|_p = \|1 - \phi(U^*)\psi(U)\|_p \leqslant \alpha$, it follows that $\|1 - T\|_p \leqslant \alpha$. Consequently $\|1 - T\| < 1$, T is invertible and W is unitary.

III. Derivations from a C^*-algebra into $L(H)$ and into the ideal K of compact operators have been studied elsewhere [2], [3], [4] and we have nothing new to add here. In this section we point out that many of the results about these derivations carry over to homomorphisms. Kadison and Ringrose [4] have shown that if $\mathfrak{A}$ is the C^*-algebra generated by an amenable group of unitary operators, then every derivation from $\mathfrak{A}$ into a dual Banach $\mathfrak{A}$-module is inner. In particular, this is true of derivations into $L(H)$. For any group G let $B(G)$ denote the Banach space of bounded functions on G with the supremum norm. A (left) invariant mean on $B(G)$ is a positive linear functional ϕ on $B(G)$ satisfying $\phi(1) = 1$ and $\phi({}_gf) = \phi(f)$ where for f in $B(G)$, g in G, ${}_gf$ is the function defined by ${}_gf(h) = f(gh)$. The group G is amenable if such a mean exists.

The following theorem generalizes a theorem of Lambert [5] since every abelian group is amenable.

THEOREM 9. *If G is an amenable group and if U_g and V_g are unitary representations of G on a Hilbert space H satisfying $\|U_g - V_g\| \leqslant \alpha < 1$ for each $g \in G$, then there is a unitary operator W on H such that $U_g = W^* V_g W$ for each g in G.*

PROOF. Let T be an operator satisfying $(Tx,y) = \phi(V_g^* U_g x,y)$ for each x and y in H, where ϕ is a left invariant mean on $B(G)$. Then

$$\begin{aligned}|((1 - T)x,y)| &= |\phi((1 - V_g^* U_g)x,y)| \leqslant \sup_{g\in G}\|1 - V_g^* U_g\|\,\|x\|\,\|y\| \\ &= \sup_{g\in G}\|V_g - U_g\|\,\|x\|\,\|y\| \leqslant \alpha\|x\|\,\|y\|.\end{aligned}$$

Therefore $\|1 - T\| < 1$ and T is invertible. Furthermore,

$$(V_g^* TU_g x,y) = \phi_h(V_g^* V_h^* U_h V_g x,y) = \phi_h(V_h^* U_h x,y) = (Tx,y)$$

so $V_g^* TU_g = T$ or $TU_g = V_g T$. Writing $T = WP$ according to its polar decomposition, W is unitary and satisfies the conclusions of the theorem.

THEOREM 10. *If a C^*-algebra $\mathfrak{A}$ is generated by an amenable subgroup of its unitary group, and if ϕ and ψ are representations of $\mathfrak{A}$ on H with $\|\phi(A) - \psi(A)\| \leqslant \alpha\|A\| < \|A\|$ for each nonzero A in $\mathfrak{A}$, then ϕ and ψ are equivalent.*

PROOF. If G is an amenable generating unitary group of $\mathfrak{A}$, then $U_g = \phi(g)$ and $V_g = \psi(g)$ are representations of G which are equivalent by Theorem 9. Consequently ϕ and ψ are equivalent.

It is certainly not the case that every derivation from a C^*-algebra into the ideal K of compact operators is inner. Johnson and Parrott [3], however, show that if $\mathfrak{A}$ is a von Neuamnn algebra which does not contain a certain kind of type II_1 factor as a direct summand, then every derivation from $\mathfrak{A}$ to K is inner. Johnson and Parrott's arguments can be modified to study ultraweakly continuous representations of von Neumann algebras which are equal modulo the ideal K. These modifications parallel those made in the proof of Theorem 1 to get Theorem 5. For example, the following can be proved:

THEOREM 12. *If $\mathfrak{A}$ is a von Neumann algebra with no type* II_1 *factor as a direct summand, and if ϕ and ψ are ultraweakly continuous representations of $\mathfrak{A}$ such that $\phi(A) - \psi(A)$ is compact for each A in $\mathfrak{A}$, then there is a partial isometry W such that $1 - W$ is compact and $W\phi(A) = \psi(A)W$ for each A in $\mathfrak{A}$.*

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