SECOND ORDER SYSTEMS WITH NONLINEAR BOUNDARY CONDITIONS

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ABSTRACT. Some existence theorems are established for nonlinear second order systems with nonlinear two point boundary conditions. The system is assumed to satisfy certain differential inequality conditions at the boundary of a region.

Let R^n be n -dimensional Euclidean space with scalar product $x \cdot y$ for x and y in R^n and norm $\|x\| = \sqrt{x \cdot x}$. In the following discussion, $f: [a, b] \times R^n \times R^n \rightarrow R^n$, where $[a, b]$ is an interval of real numbers, will be assumed continuous. The system

$$(1) \quad x'' = f(t, x, x'),$$

will be considered together with boundary conditions of the form

$$(2) \quad x(a) - A_1 x'(a) = 0,$$

$$(3) \quad x(b) + A_2 x'(b) = 0,$$

or

$$(4) \quad B_1 x(a) - x'(a) = 0,$$

$$(5) \quad B_2 x(b) + x'(b) = 0,$$

where A_1, A_2, B_1, B_2 are continuous functions from R^n to R^n .

Lasota and Yorke [3] have proved existence theorems for (1), (2), (3) in the case where A_1 and A_2 are linear and positive definite, using Lyapunov-type inequalities for f . We show in this paper that similar results are true under suitable conditions on the boundary functions even if these functions are nonlinear and differential inequality conditions on f are assumed to hold only at the boundary of a region. Nonlinear boundary conditions have been studied in the scalar case by a variety of authors, including Bebernes and Wilhelmson [1], and Jackson and Klaasen [2].

Suppose $r: R^n \rightarrow R$ is of class C^2 with gradient $u: R^n \rightarrow R^n$ and Hessian

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$P: R^n \rightarrow \mathcal{L}(R^n, R^n)$. Let the first and second derivatives of r with respect to (1) be denoted by

$$r' = u \cdot x', \quad r''_f = x'P \cdot x' + u \cdot f.$$

Auxiliary functions of this type have been used by Mawhin [4] to study (1), (2), (3) in the case $A_1 = A_2 = 0$.

We say that f satisfies condition (N) with respect to r if there is a positive, nondecreasing, continuous function ϕ defined on $(0, \infty)$ such that $s^2/\phi(s) \rightarrow \infty$ as $s \rightarrow \infty$ and $\|f(t, x, y)\| \leq \phi(\|y\|)$ for $t \in [a, b]$, $r(x) \leq 0$, $y \in R^n$. This is a Nagumo condition on f of the type used by Schmitt and Thompson [5].

THEOREM 1. *Let r be as above and assume $\{x: r(x) < 0\}$ is bounded and contains 0. Suppose $u(x) \neq 0$ whenever $r(x) = 0$ and*

$$(6) \quad \inf\{(u(x) \cdot x)/\|u(x)\| \|x\|: r(x) = 0\} = \alpha \geq 0.$$

Assume f satisfies condition (N) with respect to r and

$$(7) \quad r''_f \geq 0 \quad \text{when } r = 0 \text{ and } r' = 0.$$

Suppose $\{y: r(A_1 y) < 0\}$ is bounded and contains 0, and

$$(8) \quad \inf\{(y \cdot A_1 y)/\|y\| \|A_1 y\|: r(A_1 y) = 0\} \geq \sqrt{1 - \alpha^2},$$

$$\text{if } D = \{y: r(-A_2 y) = 0, \text{ or } r(A_1 y) = 0 \text{ and } A_2 y \neq 0\} \neq \emptyset,$$

$$(9) \quad \text{then } \inf\{(y \cdot A_2 y)/\|y\| \|A_2 y\|: y \in D\} > \sqrt{1 - \alpha^2}.$$

Then there is at least one solution x of (1), (2), (3) with $r(x(t)) \leq 0$ for $t \in [a, b]$.

PROOF. We assume initially that f satisfies

$$(7)' \quad r''_f > 0 \quad \text{when } r = 0 \text{ and } r' = 0,$$

and that initial value problems for (1) have unique solutions.

Define $W = \{(x, y) \in R^n \times R^n: x = A_1 y\}$ and $V = \{(x, y) \in W: r(x) < 0\}$. Note that $V \neq \emptyset$ since $(A_1 0, 0) \in V$. We now define a function $T: \bar{V} \rightarrow R^n$. If $(x, y) \in \bar{V}$, then there is a unique solution $x(t)$ of (1) with $(x(a), x'(a)) = (x, y)$. Since by Lemma 2.1 of [5] the derivative $x'(t)$ is bounded as long as $r(x(t)) \leq 0$, the extension theorem for ordinary differential equations implies that the trajectory $(t, x(t), x'(t))$ extends to the boundary of $[a, b] \times \{x: r(x) \leq 0\} \times R^n$. Let t_0 be the smallest value of t for which $r(x(t)) = 0$, or, if there is no such t , let $t_0 = b$. Define $T(x, y) = x(t_0) + A_2 x'(t_0)$.

We claim that T is continuous on $\bar{V}$. Let $(x, y) \in \bar{V}$ and $(x_k, y_k) \in \bar{V}$ for $k = 1, 2, \dots$, and suppose $(x_k, y_k) \rightarrow (x, y)$ as $k \rightarrow \infty$. Let x and x_k be the

solutions of (1) with $(x(a), x'(a)) = (x, y)$ and $(x_k(a), x'_k(a)) = (x_k, y_k)$ for $k = 1, 2, \dots$.

Suppose $r(x(t)) < 0$ for $t \in [a, b]$. By the standard convergence theorem, $r(x_k(t)) < 0$ for k sufficiently large and $t \in [a, b]$. Since $(x_k(b), x'_k(b)) \rightarrow (x(b), x'(b))$ as $k \rightarrow \infty$, we conclude $T(x_k, y_k) \rightarrow T(x, y)$ as $k \rightarrow \infty$ in this case.

Otherwise, there is a smallest t_0 so that $r(x(t_0)) = 0$. Suppose $a < t_0 < b$. Let $s(t) = r(x(t))$ for all t in the domain of x . Since $s(t) < 0$ for $a \leq t < t_0$ and $s(t_0) = 0$, we have $s'(t_0) \geq 0$. If $s'(t_0) > 0$, there is a $\delta > 0$ so that $s(t) > 0$ for $t_0 < t \leq t_0 + \delta$. If $s'(t_0) = 0$, then by (7) $s''(t_0) > 0$. This is impossible because $s(t) < 0$ for $a \leq t < t_0$.

Fix $\lambda \in (0, \delta)$. Let

$$\varepsilon = \inf\{|s(t)|: t \in [0, t_0 - \lambda] \cup [t_0 + \lambda, t_0 + \delta]\}.$$

Choose k sufficiently large so that $|r(x(t)) - r(x_k(t))| < \varepsilon/2$, $\|x_k(t) - x(t)\| < \lambda$ and $\|x'_k(t) - x'(t)\| < \lambda$ for $t \in [0, t_0 + \delta]$. For these values of k , $r(x_k(t)) < -\varepsilon/2 < 0$ for $t \in [0, t_0 - \lambda]$ and $r(x_k(t)) > \varepsilon/2 > 0$ for $t \in [t_0 + \lambda, t_0 + \delta]$. Hence for each such k there is a smallest t , say t_k , with $r(x_k(t_k)) = 0$ and $t_k \in (t_0 - \lambda, t_0 + \lambda)$. Furthermore, $\|x_k(t_k) - x(t_0)\|$ and $\|x'_k(t_k) - x'(t_0)\|$ are no larger than the diameter of the set

$$\{(t, x, y): t_0 - \lambda < t < t_0 + \lambda, \|x - x(t)\| < \lambda, \|y - x'(t)\| < \lambda\}.$$

Since these diameters approach 0 as $\lambda \rightarrow 0$, we conclude $T(x_k, y_k) \rightarrow T(x, y)$ as $k \rightarrow \infty$ in this case.

In case $t_0 = b$, the above argument can be modified slightly to obtain $T(x_k, y_k) \rightarrow T(x, y)$ as $k \rightarrow \infty$. Finally, consider the possibility $t_0 = a$. We have $T(x, y) = x(a) + A_2 x'(a)$ and $r(x(a)) = 0$. By (6), the angle θ_1 , between $u(x(a))$ and $x(a)$ has $\cos \theta_1 \geq \alpha$, and by (8), the angle θ_2 between $x'(a)$ and $x(a) = A_1 x'(a)$ has $\cos \theta_2 \geq \sqrt{1 - \alpha^2}$. Thus $\cos^2 \theta_1 + \cos^2 \theta_2 \geq 1$ and $\theta_1 + \theta_2 \leq \pi/2$. By a well-known result from geometry, the angle between $u(x(a))$ and $x'(a)$ is less than or equal to $\theta_1 + \theta_2$, so $u(x(a)) \cdot x'(a) \geq 0$. It follows that there is a $\delta > 0$ so that $r(x(t)) > 0$ for $a < t \leq t_0 + \delta$, and the remainder of the earlier argument can again be modified to conclude $T(x_k, y_k) \rightarrow T(x, y)$ as $k \rightarrow \infty$. Thus T is continuous on $\bar{V}$.

Define $H: R^n \rightarrow W$ by $H(y) = (A_1 y, y)$. Note that H is a homeomorphism. Let $U = H^{-1}(V)$. Then U is bounded, open, nonempty and its boundary $\partial U \subseteq \{y: r(A_1 y) = 0\}$. We will calculate the Brouwer degree for the map $T \circ H: \bar{U} \rightarrow R^n$. See Schwartz [6] for a discussion of the Brouwer degree and its properties.

Consider the maps $\lambda T \circ H + (1 - \lambda)I$ for $0 \leq \lambda \leq 1$ on $\bar{U}$, where I is the identity. Let $y \in \partial U$ and suppose $\lambda T \circ Hy + (1 - \lambda)y = 0$. If $\lambda \neq 0$, we have $T \circ Hy = -((1 - \lambda)/\lambda)y$, i.e., $(A_1 + A_2)y = -((1 - \lambda)/\lambda)y$. Note also that $r(A_1 y) = 0$ and $y \neq 0$. If $A_2 y = 0$, then $A_1 y = -((1 - \lambda)/\lambda)y$ contradicts (8). If $A_2 y \neq 0$, then (8) and (9) together imply $y \cdot (A_1 + A_2)y > 0$,

which contradicts $(A_1 + A_2)y = -((1 - \lambda)/\lambda)y$. If $\lambda = 0$, then $y = 0$, another contradiction. Thus the maps $\lambda T \circ H + (1 - \lambda)I$ do not vanish on ∂U for $0 \leq \lambda \leq 1$. The Brouwer degree $d[T \circ H, U, 0]$ exists, and by invariance of degree under homotopy, $d[T \circ H, U, 0] = d[I, U, 0] = 1$ since $0 \in U$. Thus there is a $y \in U$ so that $T \circ Hy = 0$. This means that there is a solution $x(t)$ of (1) with $(x(a), x'(a)) \in V$ and $T(x(a), x'(a)) = x(t_0) + A_2 x'(t_0) = 0$.

Suppose that $r(x(t_0)) = 0$. Since $t_0 > a$, $u(x(t_0)) \cdot x'(t_0) \geq 0$. Let ϕ_1 be the angle between $u(x(t_0))$ and $x(t_0)$, ϕ_2 the angle between $x'(t_0)$ and $A_2 x'(t_0)$ and ϕ_3 the angle between $u(x(t_0))$ and $x'(t_0)$. Then $\phi_3 \leq \pi/2$, and by (6), (9), $\cos^2 \phi_1 + \cos^2 \phi_2 > 1$, so $\phi_1 + \phi_2 < \pi/2$. Now the angle between $A_2 x'(t_0)$ and $x(t_0)$ is no larger than $\phi_1 + \phi_2 + \phi_3$, the sum of which is less than π , contradicting $A_2 x'(t_0) = -x(t_0)$. Thus $r(x(t_0)) \neq 0$, and it follows that $t_0 = b$. We conclude that x is a solution of (1), (2), (3) with $r(x(t)) < 0$ for $a \leq t \leq b$.

Now suppose initial value problems of (1) do not necessarily have unique solutions. We show first that $yP(x) \cdot y \geq 0$ whenever $r(x) = 0$ and $u(x) \cdot y = 0$. Suppose, on the contrary, that there is an x with $r(x) = 0$ and a $y \in R^n$ with $\|y\| = 1$, $u(x) \cdot y = 0$ and $yP(x) \cdot y < 0$. For all constants c , $r'(cy) = u \cdot cy = 0$, so by (7)',

$$(10) \quad r''_f = c^2 yP(x) \cdot y + u(x) \cdot f(t, x, cy) > 0.$$

Choose c large enough that $c^2/\phi(c) \geq \|u(x)\|/|yP(x) \cdot y|$. Then by condition (N),

$$|u(x) \cdot f(t, x, cy)| \leq \|u(x)\| \|f(t, x, cy)\| \leq \|u(x)\| \phi(c) \leq c^2 |yP(x) \cdot y|,$$

contradicting (10).

By Lemma 2.1 in [5], there is a $\mu > 0$ so that if $x(t)$ satisfies $\|x''\| \leq \phi(\|x'\|)$ and $r(x(t)) \leq 0$ for $t \in [a, b]$, then $\|x'(t)\| \leq \mu$ ($a \leq t \leq b$). Define for all $(t, x) \in [a, b] \times R^n$,

$$F(t, x, y) = \begin{cases} f(t, x, y), & \|y\| \leq \mu, \\ f(t, x, \mu y/\|y\|), & \|y\| > \mu. \end{cases}$$

Fix (x, y) so that $r(x) = 0$ and $r'(x, y) = 0$. If $\|y\| \leq \mu$, then $r''_F - r''_f > 0$ at (x, y) . If $\|y\| > \mu$, then

$$\begin{aligned} r''_F &= yP(x) \cdot y + u(x) \cdot f(t, x, \mu y/\|y\|) \\ &= (\|y\|^2/\mu^2)((\mu y/\|y\|)P(x) \cdot \mu y/\|y\|) + u \cdot f(t, x, \mu y/\|y\|) \\ &\geq (\mu y/\|y\|)P(x) \cdot \mu y/\|y\| + u \cdot f(t, x, \mu y/\|y\|) > 0, \end{aligned}$$

since $r''_f > 0$. Thus F satisfies (7)'.

Choose a sequence of C^1 functions $\{F_n\}_{n=1}^\infty$ which converges uniformly to F on $[a, b] \times \{x: r(x) \leq 0\} \times R^n$. For n sufficiently large, F_n satisfies (7)' and

condition (N), so we have a sequence $\{x_n\}$ so that each x_n satisfies $x'' = F_n(t, x, x')$, (2), (3) and $r(x(t)) < 0$ for $t \in [a, b]$. Some subsequence of $\{x_n\}$ converges to a solution x of $x'' = F(t, x, x')$, (2), (3) with $r(x(t)) \leq 0$ for $a \leq t \leq b$. Since F satisfies $\|F(t, x, y)\| \leq \phi(\|y\|)$ for $r(x(t)) \leq 0$ and $y \in R^d$, we have $\|x'(t)\| \leq \mu$ for $t \in [a, b]$, so x is a solution of (1).

Finally, if f satisfies (7), then $f_\epsilon = f + \epsilon u$ ($\epsilon > 0$) satisfies (7)', so the above results apply to f_ϵ , and the proof is completed by letting $\epsilon \rightarrow 0$. Q.E.D.

Of particular interest is the case that both A_1 and A_2 are linear. The assumption that $\{y: r(A_1 y) < 0\}$ is bounded requires A_1 to be nonsingular, and (8) requires, in addition, that for all $y \neq 0$, the angle between y and $A_1 y$ has cosine at least $\sqrt{1 - \alpha^2}$. Note that if $y \in \ker A_2$, then $y \notin D$. Thus (9) dictates that the cosine of the angle between y and $A_2 y$ is greater than $\sqrt{1 - \alpha^2}$ for all $y \notin \ker A_2$. The most elementary choice for r is $r(x) = \|x\|^2 - R^2$, where $R > 0$. Then (7) becomes

$$(11) \quad \|y\|^2 + x \cdot f(t, x, y) \geq 0 \quad \text{whenever } \|x\| = R \text{ and } x \cdot y = 0,$$

and (6) is satisfied with $\alpha = 1$, so one requires that A_1 be nonsingular and nonnegative definite and $A_2 y \cdot y > 0$ for all $y \notin \ker A_2$.

COROLLARY. Assume A_1 and A_2 are linear and f satisfies condition (N) with respect to $r(x) = \|x\|^2 - R^2$ for some $R > 0$. Suppose that (11) is satisfied, A_1 is nonsingular, $A_1 y \cdot y \geq 0$ for all $y \in R^n$ and $A_2 y \cdot y > 0$ for all $y \notin \ker A_2$. Then there is at least one solution x of (1), (2), (3) with $\|x(t)\| \leq R$ for $a \leq t \leq b$.

The next theorem provides criteria for the existence of a solution to (1), (4), (5). A proof can be given which is almost identical to the proof of Theorem 1.

THEOREM 2. Let r be as above and assume $\{x: r(x) < 0\}$ is bounded and contains 0. Suppose $u(x) \neq 0$ whenever $r(x) = 0$, and (6) is satisfied. Assume f satisfies condition (N) with respect to r and (7) holds. Suppose

$$(12) \quad \inf\{(x \cdot B_1 x)/\|x\| \|B_1 x\|: r(x) = 0 \text{ and } B_1 x \neq 0\} \geq \sqrt{1 - \alpha^2},$$

$$(13) \quad \begin{aligned} & B_2 x \neq 0 \text{ whenever } r(x) = 0 \text{ and} \\ & \inf\{(x \cdot B_2 x)/\|x\| \|B_2 x\|: r(x) = 0\} > \sqrt{1 - \alpha^2}. \end{aligned}$$

Then there is at least one solution x of (1), (4), (5) with $r(x(t)) \leq 0$ for $t \in [a, b]$.

REFERENCES

1. J. W. Bebernes and R. Wihelmsen, *A general boundary value problem technique*, J. Differential Equations **8** (1970), 404-415. MR **42** #575.

2. L. K. Jackson and G. Klaasen, *A variation of the topological method of Wazewski*, SIAM J. Appl. Math. **20** (1971), 124–130. MR **43** #5099.
3. A. Lasota and J. A. Yorke, *Existence of solutions of two-point boundary value problems for nonlinear systems*, J. Differential Equations **11** (1972), 509–518. MR **45** #8915.
4. J. Mawhin, *Boundary value problems for nonlinear second-order vector differential equations*, J. Differential Equations **16** (1974), 257–269. MR **50** #13696.
5. K. Schmitt and R. Thompson, *Boundary value problems for infinite systems of second-order differential equations*, J. Differential Equations **18** (1975), no. 2, 277–295. MR **51** #10794.
6. J. T. Schwartz, *Nonlinear functional analysis*, Gordon and Breach, New York, 1969.

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