AN ESTIMATE OF THE DENSITY AT THE BOUNDARY OF AN INTEGRAL CURRENT MODULO ν

SANDRA O. PAUR¹

ABSTRACT. An inequality is obtained which bounds the density at $z \in \mathbf{R}^n$ of the boundary of a $k + 1$ dimensional integral current modulo ν $(S)^\nu$ by the density of $(S)^\nu$ at z . Also, the concept of boundary tangent developed in [3] is shown to be in agreement with Federer's concept of a measuretheoretic exterior normal if $\nu = 0$ and S is obtained by integration over an $\mathcal{L}^n$ measurable subset of $\mathbf{R}^n$.

Introduction. In [3, 5.1] a boundary tangent k vector at $z \in \mathbf{R}^n$ to a $k + 1$ dimensional integral current modulo ν $(S)^\nu$ was defined geometrically in terms of the intersection index of $(S)^\nu$ with a sufficiently small sphere centered at z and an oriented $n - k$ plane orthogonal to the proposed tangent k vector. Using this definition and results obtained in [3], we prove in this note the density inequality 2.3:

$$\theta^k(\|\partial S\|^\nu, z) \leq 2\theta_*^{k+1}(\|S\|^\nu, z)$$

for $\mathcal{H}^k$ almost all $z \in \mathbf{R}^n$. For the case where $\nu = 0$ and S is the n -current obtained by integration over an $\mathcal{L}^n$ measurable subset of $\mathbf{R}^n$, the inequality follows from [2, 4.5.6]. The inequality for $\nu = 0$ and arbitrary k was obtained by Brothers [1, 3.6]; however, the inequality for $\nu > 0$ is also interesting because an integral current modulo ν need have no integral current representative [3, 2.5].

We also show that our concept of boundary tangent agrees with Federer's concept of a measuretheoretic exterior normal $\mathbf{n}(A, z)$ [2, 4.5.5., 4.5.6(2)] in the case where $\nu = 0$ and S is obtained by integration over an $\mathcal{L}^n$ measurable subset A of $\mathbf{R}^n$.

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1. The notation used is consistent with [2, pp. 669–671] and [3]. Let k be a fixed integer, $0 < k < n$. P (or P_1) will denote an oriented k plane in $\mathbf{R}^n$, and

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the oriented $n - k$ plane orthogonal to P through $\omega \in P$ will be denoted by $P^\perp(\omega)$ oriented so that $P^\perp(\omega)^\rightarrow = *P^\rightarrow$. The orthogonal projection of $\mathbf{R}^n$ onto P will be represented by π_P .

If P is such that $\text{spt } P = \{x: x_{k+1} = \dots = x_n = 0\}$, then the "folding" map $F: \mathbf{R}^n \rightarrow P \times \mathbf{R}^+$, $\mathbf{R}^+ = \{x \in \mathbf{R}: x \geq 0\}$, is defined by

$$F(x) = \left(x_1, \dots, x_k, (x_{k+1}^2 + \dots + x_n^2)^{1/2}\right)$$

for $x = (x_1, \dots, x_n) \in \mathbf{R}^n$.

If P is an oriented $n - 1$ plane, $y \in \mathbf{R}^n$ and $\omega = \pi_P(y)$, then

$$\pi_P^{-1}(\omega) \cap \mu_{|y|} \mathbf{S}^{n-1} = \{y, y'\}.$$

Set $\sigma_P(y) = y'$. If $\text{spt } P = \{x: x_n = 0\}$, denote $\sigma_P(y)$ by $\sigma(y)$ and π_P by π .

2. If $S \in \mathcal{R}_n(\mathbf{R}^n)$ then $S = f \llcorner \mathbf{E}^n$ where f is an integer-valued $\mathcal{L}^n$ summable function. If $\tau(S, z) = -\theta P^\rightarrow$ exists (see [3, 5.1]) and $P^\rightarrow = *e_n$, then for almost all $(t, \omega) \in \mathbf{R}^+ \times P$, $I(S, z, t, P^\perp(\omega)) = f(x) - f \circ \sigma(x)$ (see [3, §4]), where $|x - z| = t$, $(x - z) \cdot e_n > 0$ and $\pi(x) = \omega$.

2.1. LEMMA. If $S \in \mathcal{R}_n(\mathbf{R}^n)$ and $z \in \mathbf{R}^n$ are such that there are $\theta > 0$ and an $n - 1$ plane P satisfying [3, 5.1(1)], and if $v = (-1)^n * P^\rightarrow$, then

$$\theta^n [\mathcal{L}^n \llcorner \{x: f(x) - f \circ \sigma(x) \neq \theta, x \cdot v < 0\}, 0] = 0.$$

PROOF. Assume $z = 0$ and $P^\rightarrow = *e_n$. Then $v = -e_n$. Let

$$\beta = \{x: f(x) - f \circ \sigma(x) \neq \theta, x \cdot e_n > 0\}.$$

Suppose $\theta^{*n} [\mathcal{L}^n \llcorner \beta, 0] > 0$. Then by the coarea formula [2, 3.2.11],

$$(*) \quad 0 < \int_0^1 \limsup_{\rho \rightarrow 0^+} \rho^{-n+1} \mathcal{H}^{n-1}(\beta \cap \mu_{t\rho} \mathbf{S}^{n-1}) d\mathcal{L}^1 t < \infty.$$

However, from $(*)$ and because θ and P satisfy [3, 5.1(1)], we can choose $t \in (0, 1)$ and a sequence of positive numbers $r_1, r_2, \dots$ converging to zero so that

$$(i) \quad \lim_{j \rightarrow \infty} r_j^{-n+1} \mathcal{H}^{n-1}(\beta \cap \mu_{tr_j} \mathbf{S}^{n-1}) > 0,$$

and so that

$$(ii) \quad \lim_{j \rightarrow \infty} r_j^{-n+1} \int_{P \cap B(0, r_j t)} |I[S, 0, tr_j, P^\perp(\omega)] - \theta| d\mathcal{L}^{n-1} \omega = 0.$$

From (ii) it follows that

$$r_j^{-n+1} \mathcal{L}^{n-1}(A_j) \rightarrow 0 \quad \text{as } j \rightarrow \infty,$$

where $A_j = \pi(\beta \cap \mu_{tr_j} \mathbf{S}^{n-1})$. Denote $E_j = \mu_{1/r_j}(\beta \cap \mu_{tr_j} \mathbf{S}^{n-1}) \subset \mathbf{S}^{n-1}$. Then $\pi(E_j) = \mu_{1/r_j}(A_j)$, so $\mathcal{L}^{n-1}(\pi(E_j)) = \int_{E_j} x \cdot e_n d\mathcal{H}^{n-1} x \rightarrow 0$ as $j \rightarrow \infty$. However, from (i), $\mathcal{H}^{n-1}(E_j) \rightarrow a > 0$ as $j \rightarrow \infty$, which is a contradiction since $x \cdot e_n > 0$.

2.2. THEOREM. Assume $\nu > 0$. If $(S)^\nu \in \mathcal{R}_{k+1}^\nu(\mathbf{R}^n)$ and $z \in \mathbf{R}^n$ is such that $\theta > 0$ and a k plane P exist which satisfy [3, 5.1(1)], then $\theta \leq 2\theta_*^{k+1}(\|S\|^\nu, z)$.

PROOF. Assume $z = 0$ and $P^\rightarrow = -e_1 \wedge \cdots \wedge e_k$. Suppose

$$T \in \mathcal{R}_{k+1}(\mathbf{R}^n)$$

is representative modulo ν for $F_*(S)^\nu \in \mathcal{R}_{k+1}^\nu(\mathbf{R}^{k+1})$. Then by [2, 4.3.2(7)] and the fact that $\text{spt } T \subset P \times \mathbf{R}^+$,

$$I[T, 0, t, P^\perp(\omega)] = I[(S)^\nu, 0, t, P^\perp(\omega)]$$

for $\mathbb{L}^1 \times \mathbb{L}^k$ almost all $(t, \omega) \in (0, 1) \times P$. Thus θ and P also satisfy [3, 5.1(1)] for T . Denote $T = g \llcorner \mathbf{E}^{k+1}$ where g is a bounded, integer-valued, $\mathbb{L}^{k+1}$ summable function. Thus by the lemma,

$$\lim_{\rho \rightarrow 0^+} \rho^{-k-1} \int_{\{x: |x| < \rho, x \cdot e_{k+1} > 0\}} |g(x) - \theta| d\mathbb{L}^{k+1}x = 0.$$

Hence,

$$\begin{aligned} \theta \frac{\alpha(k+1)}{2} &= \lim_{\rho \rightarrow 0^+} \rho^{-k-1} \int_{\{x: |x| < \rho, x \cdot e_{k+1} > 0\}} g(x) d\mathbb{L}^{k+1}x \\ &= \lim_{\rho \rightarrow 0^+} \rho^{-k-1} \int_{B(0, \rho)} g(x) d\mathbb{L}^{k+1}x \\ &= \alpha(k+1)\theta^{k+1}(\|T\|^\nu, 0) \leq \alpha(k+1)\theta_*^{k+1}(\|S\|^\nu, 0) \end{aligned}$$

by [2, p. 371] with $\sigma(x) = 1$.

2.3. COROLLARY. If $(S)^\nu \in \mathcal{I}_{k+1}^\nu(\mathbf{R}^n)$, then for $\mathcal{H}^k$ almost all $z \in \mathbf{R}^n$,

$$\theta^k(\|\partial S\|^\nu, z) \leq 2\theta_*^{k+1}(\|S\|^\nu, z).$$

PROOF. By [3, 5.7(1) and (2)], $\theta^k(\|\partial S\|^\nu, z) = |\tau((S)^\nu, z)|$ (see [3, 5.1]) for $\mathcal{H}^k$ almost all $z \in \mathbf{R}^n$.

2.4. THEOREM. If f is the characteristic function of a bounded $\mathbb{L}^n$ measurable subset A of $\mathbf{R}^n$, $z \in \mathbf{R}^n$, and either $\mathbf{n}(A, z)$ exists and $|\mathbf{n}(A, z)| = 1$ or $\tau(S, z)$ exists and $|\tau(S, z)| > 0$ where $S = \mathbf{E}^n \llcorner A$, then $*\mathbf{n}(A, z) = \tau(S, z)$.

PROOF. Assume $z = 0$ and suppose $\tau(S, 0)$ exists. Assume

$$\tau(S, 0) = -\theta * e_n, \quad \theta = |\tau(S, 0)| > 0.$$

Then for each $r > 0$ and for $\mathbb{L}^1 \times \mathbb{L}^{n-1}$ almost all $(t, \omega) \in (0, 1) \times P \cap B(0, rt)$, $I[S, 0, rt, P^\perp(\omega)] = f(x) - f \circ \sigma(x)$ where $x \cdot e_n > 0$ and $\pi^{-1}\{\omega\} \cap \mu_{rt} S^{n-1} = \{x, \sigma(x)\}$.

$$0 \leq |I[S, 0, rt, P^\perp(\omega)]| = |f(x) - f \circ \sigma(x)| \leq 1$$

so $\theta = 1$ since θ is a positive integer. By 2.1,

$$\theta^n[\mathbb{L}^n \llcorner \{x: f(x) = 0, x \cdot e_n > 0\} \cup \{x: f(x) = 1, x \cdot e_n < 0\}, 0] = 0.$$

Thus $\mathbf{n}(A, z) = -e_n$ so $*\mathbf{n}(A, z) = \tau(S, z)$.

Suppose $\mathbf{n}(A, 0)$ exists and $|\mathbf{n}(A, 0)| = 1$. Assume $\mathbf{n}(A, 0) = -e_n$. Choose $\theta = 1$ and let P be the $n-1$ plane such that $0 \in P$ and $*e_n = P^\rightarrow$. If $\omega \in P \cap B(0, rt)$ then there is $x \in \mathbf{R}^n$ so that $x \cdot e_n > 0$ and

$$\pi^{-1}\{\omega\} \cap \mu_{rt}S^{n-1} = \{x, \sigma(x)\}.$$

By [2, 3.2.12]

$$\begin{aligned} & \int_0^1 \int_{P \cap B(0,rt)} r^{-n+1} |I[S, 0, rt, P^\perp(\omega)] - 1| d\mathcal{L}^{n-1} d\mathcal{L}^1 t \\ & \leq \int_0^1 \int_{\{x: |x|=rt, x \cdot e_n > 0\}} r^{-n+1} |f(x) - 1| d\mathcal{H}^{n-1} x d\mathcal{L}^1 t \\ & \quad + \int_0^1 \int_{\{x: |x|=rt, x \cdot e_n < 0\}} r^{-n+1} |f(x)| d\mathcal{H}^{n-1} x d\mathcal{L}^1 t \\ & \leq \int_{\{x: x \cdot e_n > 0, |x| < r\}} r^{-n} |f(x) - 1| d\mathcal{L}^n x + \int_{\{x: x \cdot e_n < 0, |x| < r\}} r^{-n} |f(x)| d\mathcal{L}^n x \end{aligned}$$

which has limit zero as $r \rightarrow 0^+$ by the definition of $n(A, 0)$. Thus (1) of the definition of $\tau(S, 0)$ is satisfied. Suppose P_1 is an oriented $n-1$ dimensional plane such that $0 \in P_1$ and $P_1 \neq \pm P$. Choose $v \in P_1 \cap S^{n-1}$ so that $v \notin \text{spt } P$ and $q \in \mathbb{R}$ such that $V(0, v, q) \cap P = \emptyset$. For $\mathcal{L}^1$ almost all

$$t \in (0, 1),$$

for $\mathcal{L}^{n-1}$ almost all $\omega \in \pi_{P_1}[V(0, v, q) \cap \mu_{rt}S^{n-1}]$,

$$|I[S \llcorner V(0, v, q), 0, rt, P_1^\perp(\omega)]| = |f(y) - f \circ \sigma_{P_1}(y)|$$

where $y \in \pi_{P_1}^{-1}\{\omega\} \cap \mu_{rt}S^{n-1}$. If $\omega \cdot e_n < 0$ then $y \cdot e_n < 0$ and

$$\sigma_{P_1}(y) \cdot e_n < 0.$$

If $\omega \cdot e_n > 0$ then $y \cdot e_n > 0$ and $\sigma_{P_1}(y) \cdot e_n > 0$. Thus by [2, 3.2.12],

$$\begin{aligned} & \limsup_{r \rightarrow 0^+} \int_0^1 \int_{P_1} r^{-n+1} |I[S \llcorner V(0, v, q), 0, rt, P_1^\perp(\omega)]| d\mathcal{L}^{n-1} \omega d\mathcal{L}^1 t \\ & \leq \limsup_{r \rightarrow 0^+} \int_{\{y: |y| < r, y \cdot e_n < 0\}} r^{-n} |f(y) - f \circ \sigma_{P_1}(y)| d\mathcal{L}^n y \\ & \quad + \limsup_{r \rightarrow 0^+} \int_{\{y: |y| < r, y \cdot e_n > 0\}} r^{-n} |f(y) - f \circ \sigma_{P_1}(y)| d\mathcal{L}^n y \\ & \leq 2 \limsup_{r \rightarrow 0^+} \int_{\{y: |y| < r, y \cdot e_n < 0\}} r^{-n} |f(y)| d\mathcal{L}^n y \\ & \quad + 2 \limsup_{r \rightarrow 0^+} \int_{\{y: |y| < r, y \cdot e_n > 0\}} r^{-n} |f(y) - 1| d\mathcal{L}^n y, \end{aligned}$$

which is zero by the definition of $n(A, 0)$. Thus (2) of the definition of $\tau(S, 0)$ is satisfied.

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DEPARTMENT OF MATHEMATICS, NORTH CAROLINA STATE UNIVERSITY, RALEIGH, NORTH CAROLINA 27607