A SERIES AND ITS ASSOCIATED CONTINUED FRACTION¹

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ABSTRACT. Let $\alpha = (1 + \sqrt{5})/2$. In this paper it is proved that

$$T(\alpha) \equiv \sum_{r=1}^{\infty} \frac{1}{2^{[r\alpha]}} = 1 \cfrac{1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{8 + \ddots + \cfrac{1}{t_n + \ddots}}}}}}{1}$$

where $t_n = 2^{f_n-2}$ and (f_n) is the Fibonacci sequence. It is also shown that $T(\alpha)$ is transcendental.

1. Introduction. If x is a real number ≥ 1 we may form the series $T(x) = \sum_{r=1}^{\infty} 1/2^{[rx]}$. It is easy to see that the series is convergent and that $0 < T(x) \leq 1$.

In this paper we shall obtain a remarkable continued fraction expansion for $T(\alpha)$ where $\alpha = (1 + \sqrt{5})/2$. Furthermore, we show that $T(\alpha)$ is transcendental.

Recall that the Fibonacci sequence (f_n) can be defined by $f_0 = 1$; $f_1 = 1$; $f_n = f_{n-1} + f_{n-2}$ ($n \geq 2$) and that α has the continued fraction expansion $[1, 1, 1, \dots]$ with convergents f_n/f_{n-1} for $n \geq 1$. The notation and results on continued fractions used here are to be found in [2].

In §3 we shall prove

THEOREM 1. $T(\alpha) = [0, 1, 2, 2, 4, 8, 32, 256, \dots, t_n, \dots]$ where $t_0 = 1$, $t_1 = 1$, $t_n = 2^{f_n-2}$ ($n \geq 2$).

2. Preliminary lemmas. Let m be an arbitrary positive integer. Let $p = f_{2m-2}$, $q = f_{2m-1}$, $p' = f_{2m}$, $q' = f_{2m+1}$. Then q/p and q'/p' are consecutive even convergents to α . Clearly $p' = p + q$ and $q' = p' + q$. The identities (i)

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$q'p - p'q = 1$ and (ii) $qq' = p'^2 - 1$ are well known and can be easily proved by induction (on m).

For each integer $r \geq 0$ let $a_r = [r\alpha]$. The sequence begins as 1, 3, 4, 6, 8, 9, 11, 12,

LEMMA 1. *With p, q, p', q' as defined above we have:*

- (a) $[rq/p] = a_r$ for $0 \leq r \leq p$,
- (b) $[rq'/p'] = [rq/p]$ for $0 \leq r < p'$.

PROOF. (a) It is known that $0 < \alpha - q/p < 1/p^2$. Thus $rq/p < r\alpha < rq/p + r/p^2 \leq (rq + 1)/p$. If $rq/p < a_r$ then $rq < pa_r$ so that $rq + 1 \leq pa_r$ and hence, $(rq + 1)/p \leq a_r$. Then $a_r < r\alpha < (rq + 1)/p \leq a_r$, which is a contradiction. Thus $[rq/p] = a_r$.

(b) Since $q'/p' = q/p + 1/pp'$ it follows that $rq'/p' = rq/p + r/pp' < rq/p + 1/p$ if $0 \leq r < p'$. By reasoning as in (a) we conclude that $[rq'/p'] = [rq/p]$.

REMARK. Using (a) and (b) from Lemma 1, we can sharpen (a) to read: $[rq/p] = a_r$ for $0 \leq r < p'$.

LEMMA 2. *With p, q, p', q' as defined above we have*

- (a) $a_{p+r} = a_p + a_r = q + a_r$, $0 \leq r < q$,
- (b) $a_{q+r} = p' + a_r$, $1 \leq r < p'$.

PROOF. (a) If $0 \leq r < q$ then $p + r < p + q = p'$ so we may apply the Remark to conclude that $a_{p+r} = [(p + r)q/p] = [q + rq/p] = q + a_r$.

(b) Since $a_{q+r} = [(q + r)q'/p'] = [(qq' + rq')/p']$ and $qq' = p'^2 - 1$, it follows that $a_{q+r} = p' + [(rq' - 1)/p']$.

Note that this shows that $a_q = p' - 1$. Now $(p', q') = 1$ so if $1 \leq r < p'$, then rq'/p' is not an integer and, hence, $[(rq' - 1)/p'] = [rq'/p'] = a_r$. Thus $a_{q+r} = p' + a_r$.

3. The main theorem. Let S be the value of the continued fraction $[0, 1, 2, 2, 4, 8, 32, 256, \dots, t_n, \dots]$ where $t_0 = 0$, $t_1 = 1$ and $t_n = 2^{f_{n-2}}$ ($n \geq 2$). Denote the n th convergent to S by p_n/q_n . Then it is clear that

$$p_0 = 0; p_1 = 1; p_2 = 2; p_3 = 5, \dots, p_n = 2^{f_{n-1}}p_{n-1} + p_{n-2} \quad (n \geq 2),$$

$$q_0 = 1; q_1 = 1; q_2 = 3; q_3 = 7, \dots, q_n = 2^{f_{n-2}}q_{n-1} + q_{n-2} \quad (n \geq 2).$$

The first six convergents to S are $0, 1, \frac{2}{3}, \frac{5}{7}, \frac{22}{31}, \frac{181}{255}$. It is easy to prove by induction that $q_n = 2^{f_n} - 1$ ($n \geq 0$). From general results on continued fractions we have

$$(1) \quad p_{2m-1}/q_{2m-1} - p_{2m+1}/q_{2m+1} = 2^{f_{2m-1}}/q_{2m-1}q_{2m+1}.$$

If m, n are positive integers with $n \geq m$ then define

$$T(m, n) = \sum_{r=1}^{\infty} \frac{1}{2^{\lfloor rn/m \rfloor}}.$$

$T(m, n)$ is a rational number. By some computation it is found that $T(1, 1) = 1$; $T(2, 3) = \frac{5}{7}$; $T(5, 8) = \frac{181}{255}$. These numbers occur as odd

convergent to S . In fact, in general we have

LEMMA 3. $T(f_{2m}, f_{2m+1}) = p_{2m+1}/q_{2m+1}$ for $m \geq 0$.

PROOF. Since $T(1, 1) = 1 = p_1/q_1$ it suffices to show that

$$T(f_{2m-2}, f_{2m-1}) - T(f_{2m}, f_{2m+1}) = 2^{f_{2m-1}} / (2^{f_{2m-1}} - 1)(2^{f_{2m+1}} - 1)$$

(this is the analogue of (1)).

Again let $p = f_{2m-2}$, $q = f_{2m-1}$, $p' = f_{2m}$ and $q' = f_{2m+1}$.

$$\begin{aligned} T(p, q) &= \sum_{r=1}^{\infty} \frac{1}{2^{\lfloor rq/p \rfloor}} = \frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \cdots + \frac{1}{2^q} \\ &\quad + \frac{1}{2^{a_1+q}} + \cdots + \frac{1}{2^{2q}} + \cdots \\ &= (1/2^{a_1} + 1/2^{a_2} + \cdots + 1/2^q)(2^q / (2^q - 1)) \\ &= (2^{q-a_1} + \cdots + 2^{q-a_{q-1}} + 1) / (2^q - 1). \end{aligned}$$

Similarly

$$T(p', q') = \frac{(2^{q'-a_1} + \cdots + 2^{q'-a_{q'-1}} + 1)}{2^{q'} - 1}.$$

Hence

$$\begin{aligned} T(p, q) - T(p', q') &= \frac{[(2^{q'} - 1)(2^{q-a_1} + \cdots + 2^{q-a_{q-1}} + 1) - (2^q - 1)(2^{q'-a_1} + \cdots + 2^{q'-a_{q'-1}} + 1)]}{(2^q - 1)(2^{q'} - 1)}. \end{aligned}$$

The numerator simplifies to

$$\begin{aligned} &-(2^{q-a_1} + 2^{q-a_2} + \cdots + 1) - (2^{q+q'-a_{q+1}} + \cdots + 2^q) \\ &\quad + (2^{q'-a_1} + \cdots + 1). \end{aligned}$$

Now, by Lemma 2, $a_{p+r} = q + a_r$ and $a_{q+r} = p' + a_r$. Thus we have $q + q' - a_{p+r} = q' - a_r$ ($0 \leq r < q$) and $q - a_r = q' - a_{p'-p+r}$ ($1 \leq r \leq p$). Hence the numerator becomes $-2^q + 2^{q'-a_{q-p}} = -2^q + 2^{q+1} = 2^q$ and $T(p, q) - T(p', q') = 2^q / (2^q - 1)(2^{q'} - 1)$.

PROOF OF THEOREM 1. Lemma 1(a) shows that $|T(\alpha) - T(p, q)| < 1/2^q$ since both series agree up to and including the term $1/2^q$. Thus

$$T(f_{2m}, f_{2m+1}) \rightarrow T(\alpha) \text{ as } m \rightarrow \infty.$$

But we have $p_{2m+1}/q_{2m+1} \rightarrow S$ as $m \rightarrow \infty$. By Lemma 3, it follows that $S = T(\alpha)$.

THEOREM 2. $T(\alpha)$ is transcendental. In fact the inequality $|T(\alpha) - a/b| < 1/b^{2.6}$ is true for an infinity of positive integer pairs a, b .

PROOF. If p, q are as in Lemma 3 then, using the Remark, it is easy to see that

$$0 \leq T(p, q) - T(\alpha) \leq (1/2^{q'-1} + 1/2^{q'} + \dots) = 1/2^{q'-2}.$$

Now $T(p, q) = a/b$ where $b = 2^q - 1$. Also $q'/q \rightarrow (3 + \sqrt{5})/2$ as $m \rightarrow \infty$. Thus $q' > 2.6q + 2$ for large enough q and, hence, $0 < T(p, q) - T(\alpha) < 1/b^{2.6}$. By Roth's Theorem ([1] or [3]) we conclude that $T(\alpha)$ is transcendental.

REFERENCES

1. J. W. S. Cassels, *An introduction to Diophantine approximation*, Cambridge Tracts in Math. and Math. Phys., no. 45, Cambridge Univ. Press, London and New York, 1957. MR 19, 396.
2. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, 3rd ed., Clarendon Press, Oxford, 1954. MR 16, 673.
3. K. F. Roth, *Rational approximations to algebraic numbers*, Mathematika 2 (1955), 1-20; Corrigendum, 168. MR 17, 242.

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