A GENERALIZATION OF A THEOREM OF S. N. BERNSTEIN¹

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ABSTRACT. A rational approximation scheme is exhibited for a set Δ which consists of a finite union of compact subintervals of the real line. This rational approximation scheme provides a characterization of the analytic functions on Δ which generalizes S. N. Bernstein's characterization of the analytic functions on $[-1, 1]$.

1. Bernstein's theorem. Let E be a compact subset of the real line R . A complex-valued function f defined on E is said to be *analytic* on E if f is the restriction to E of a holomorphic function F defined on an open subset Ω of the complex plane C that contains E . Note that Ω need not be a region.

A theorem of S. N. Bernstein relates the analyticity of a function f defined on the interval $I = [-1, 1]$ to the rate by which f may be uniformly approximated by polynomials on I . In order to state Bernstein's theorem precisely, let P_n be the class of all polynomials of degree $\leq n$, let

$$\|g\|_I = \sup\{|g(x)| : x \in I\}$$

for any function g defined on I , and let

$$e(f, P_n, I) = \inf\{\|f - p\|_I : p \in P_n\} \quad (n = 1, 2, 3, \dots).$$

Let D_ρ be the open elliptical region defined by

$$(1) \quad D_\rho = \{z \in C : |z + 1| + |z - 1| < \rho + 1/\rho\},$$

for $0 < \rho < 1$.

THEOREM 1 (S. N. BERNSTEIN). *Let f be a complex-valued function defined on $I = [-1, 1]$. Then*

$$(2) \quad \limsup_{n \rightarrow \infty} \sqrt[n]{e(f, P_n, I)} \leq \rho,$$

with $0 < \rho < 1$, if and only if f is the restriction to I of a holomorphic function F defined on the region D_ρ . In particular,

$$(3) \quad \lim_{n \rightarrow \infty} \sqrt[n]{e(f, P_n, I)} = 0$$

if and only if the holomorphic function F is entire.

PROOF. See [1, pp. 110–113].

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Suppose one replaces the interval I by a union of intervals

$$\Delta = \bigcup_{j=0}^r [a_j, b_j],$$

where $-\infty < a_0 < b_0 < a_1 < \cdots < b_r < +\infty$, and seeks a similar characterization of the analytic functions on Δ . If one thinks of polynomials as rational functions with poles at the point $\lambda_0 = \infty$, then one might try to generalize Bernstein's theorem to Δ by replacing the class of polynomials by a class of rational functions with poles at certain points $\lambda_1, \dots, \lambda_r$ between each of the intervals $[a_0, b_0], [a_1, b_1], \dots, [a_r, b_r]$, as well as at the point $\lambda_0 = \infty$. The difficulty in finding such a generalization is choosing the poles $\lambda_1, \dots, \lambda_r$.

In this note the work of Rosenblum and Rovnyak on Cayley inner functions (see [3]) is used to develop a rational approximation scheme for Δ that corresponds to polynomial approximation on I and provides a characterization of the analytic functions on Δ analogous to that in Bernstein's theorem.

2. Let $\Delta = \bigcup_{j=0}^r [a_j, b_j]$, where $-\infty < a_0 < b_0 < a_1 < \cdots < b_r < +\infty$, and let ϕ be the rational function

$$\phi(z) = \left(b - a \prod_{j=0}^r \frac{b_j - z}{a_j - z} \right) / \left(1 - \prod_{j=0}^r \frac{b_j - z}{a_j - z} \right).$$

The poles of $\phi(z)$ are the solutions of the equation $1 = \prod_{j=0}^r (b_j - z)/(a_j - z)$, and hence are at the points $z = \infty$ and $z = \lambda_1, \dots, \lambda_r$, where $b_{j-1} < \lambda_j < a_j$ ($j = 1, \dots, r$). Moreover,

$$\phi^{-1}([a, b]) = \Delta \quad \text{and} \quad \phi^{-1}([a, b]^c) = \Delta^c \setminus \{\lambda_1, \dots, \lambda_r\}$$

(where $E^c = \{x \in R: x \notin E\}$ whenever $E \subseteq R$), and ϕ is a strictly increasing mapping of each of the intervals $[a_0, b_0], [a_1, b_1], \dots, [a_r, b_r]$ onto $[a, b]$. The rational function ϕ is an example of a *Cayley inner function* and is said to *adapt* Δ to $[a, b]$; see [3] for details.

Let $\lambda_0 = -\infty, \lambda_{r+1} = +\infty$, and put

$$\phi_j = \phi|(\lambda_j, \lambda_{j+1}) \quad (j = 0, 1, \dots, r).$$

Define the functions $k_0, k_1, \dots, k_r$ on Δ by

$$k_0(t) = 1, \quad k_1(t) = (t - \lambda_1)^{-1}, \dots, k_r(t) = (t - \lambda_r)^{-1}.$$

(We use t as a dummy variable for functions defined on Δ and x for functions defined on $[a, b]$.) If E is a compact subset of the real line R , then $C(E)$ is the Banach space of all complex-valued continuous functions on E with norm $\|f\|_E = \sup\{|f(x)|: x \in E\}$; $C(E)^{r+1}$ is the direct sum of $r+1$ copies of $C(E)$ and is a Banach space under any of several equivalent norms.

LEMMA. Define the linear operator $B: C([a, b])^{r+1} \rightarrow C(\Delta)$ by

$$(Bg)(t) = \sum_{j=0}^r k_j(t) g_j(\phi(t)),$$

where $g = (g_0, g_1, \dots, g_r)$. Then B is bounded and has a bounded inverse defined by

$$(B^{-1}f)(x) = A(x)^{-1} \cdot \begin{bmatrix} f(\phi_0^{-1}(x)) \\ \vdots \\ f(\phi_r^{-1}(x)) \end{bmatrix}, \quad (f \in C(\Delta)),$$

where $A(x)$ is the matrix

$$A(x) = [a_{ij}(x)]_{i,j=0}^r = [k_j(\phi_i^{-1}(x))]_{i,j=0}^r.$$

PROOF. If $g = (g_0, g_1, \dots, g_r) \in C([a, b])^{r+1}$, then

$$\|Bg\|_{\Delta} \leq (r+1) \cdot K \cdot \max_j \|g_j \circ \phi\|_{\Delta} = (r+1) \cdot K \cdot \max_j \|g_j\|_{[a,b]},$$

where $K = \max_j \|k_j\|_{\Delta} < +\infty$. Hence B is a bounded operator.

If $f \in C(\Delta)$, then $f = Bg$ with $g = (g_0, g_1, \dots, g_r)$ if and only if

$$f(\phi_i^{-1}(x)) = \sum_{j=0}^r k_j(\phi_i^{-1}(x)) g_j(x) \quad (i = 0, 1, \dots, r);$$

that is, if and only if

$$f(\phi_i^{-1}(x)) = \sum_{j=0}^r a_{ij}(x) g_j(x) \quad (i = 0, 1, \dots, r),$$

where $a_{ij}(x) = k_j(\phi_i^{-1}(x))$ for $i, j = 0, 1, \dots, r$. The matrix $A(x) = [a_{ij}(x)]$ has the form

$$\begin{bmatrix} 1, & \frac{1}{\eta_0 - \lambda_1}, & \dots, & \frac{1}{\eta_0 - \lambda_r} \\ \dots & & & \\ 1, & \frac{1}{\eta_r - \lambda_1}, & \dots, & \frac{1}{\eta_r - \lambda_r} \end{bmatrix},$$

where $\eta_i = \phi_i^{-1}(x)$. Since

$$\det \left[\frac{1}{\eta_i - \lambda_j} \right]_{i,j=0}^r = \frac{\prod_{r \geq i > j \geq 0} (\eta_i - \eta_j)(\lambda_j - \lambda_i)}{\prod_{i,j=0, \dots, r} (\eta_i - \lambda_j)}$$

(see [2, part seven, Problem 3]), we deduce that

$$\det A(x) = \lim_{\lambda_0 \rightarrow \infty} \left(-\lambda_0 \det \left[\frac{1}{\eta_i - \lambda_j} \right]_{i,j=0}^r \right) \neq 0$$

for $a \leq x \leq b$. Thus $A(x)$ is invertible for $a \leq x \leq b$, and

$$A(x)^{-1} = [d_{ij}(x)], \quad d_{ij}(x) = C_{ji}(x)/\det A(x),$$

where $C_{ij}(x)$ is the cofactor of $a_{ij}(x)$ in $A(x)$. It follows that if f is in $C(\Delta)$ and if

$$g(x) = \begin{bmatrix} g_0(x) \\ g_1(x) \\ \vdots \\ g_r(x) \end{bmatrix} = A(x)^{-1} \cdot \begin{bmatrix} f(\phi_0^{-1}(x)) \\ f(\phi_1^{-1}(x)) \\ \vdots \\ f(\phi_r^{-1}(x)) \end{bmatrix},$$

then $g \in C([a, b])^{r+1}$, $f = Bg$, and $\|g_i\|_{[a,b]} \leq M\|f\|_\Delta$ ($i = 0, 1, \dots, r$) for some finite constant M independent of f . This completes the proof.

Since the matrix $A(x) = [a_{ij}(x)]$ has analytic entries, it follows that, if $f = Bg$ and $g = (g_0, g_1, \dots, g_r)$, then f is analytic on Δ if and only if $g_0, g_1, \dots, g_r$ are analytic on $[a, b]$. For $n = 1, 2, 3, \dots$, let K_n be the complex vector space spanned by the functions

$$1, t, \dots, t^n; \frac{1}{t - \lambda_1}, \dots, \frac{1}{(t - \lambda_1)^{n+1}}; \dots; \frac{1}{t - \lambda_r}, \dots, \frac{1}{(t - \lambda_r)^{n+1}}.$$

Then K_n is the image under B of all $(r+1)$ -tuples of polynomials of degree $\leq n$; see [3, Lemma 2.4]. The class K_n will replace the class of polynomials P_n in our generalization of Bernstein's theorem to Δ .

3. A generalization of Bernstein's theorem. For simplicity, we let $[a, b]$ be the interval $I = [-1, 1]$. For each complex-valued function f defined on Δ , put

$$e(f, K_n, \Delta) = \inf\{\|f - k\|_\Delta; k \in K_n\} \quad \text{for } n = 1, 2, 3, \dots$$

THEOREM 2. *A complex-valued function f defined on Δ is analytic on Δ if and only if*

$$(4) \quad \limsup_{n \rightarrow \infty} \sqrt[n]{e(f, K_n, \Delta)} < 1,$$

and

$$(5) \quad \lim_{n \rightarrow \infty} \sqrt[n]{e(f, K_n, \Delta)} = 0$$

if and only if f has a holomorphic extension F to the region $\Omega_\phi = \{z \in \mathbb{C}; z \neq \lambda_1, \dots, z \neq \lambda_r\}$. There exists a real number ρ that depends only on ϕ , with $0 < \rho < 1$, such that f has a holomorphic extension F to a region if

$$(6) \quad \limsup_{n \rightarrow \infty} \sqrt[n]{e(f, K_n, \Delta)} < \rho.$$

PROOF. Let f be a function on Δ with

$$\limsup_{n \rightarrow \infty} \sqrt[n]{e(f, K_n, \Delta)} < \sigma < 1.$$

Then there exists an integer N such that $e(f, K_n, \Delta) < \sigma^n$ if $n \geq N$. Hence f is the uniform limit of continuous functions on Δ and so $f \in C(\Delta)$. By the Lemma, there exist continuous functions $g_0, g_1, \dots, g_r$ on I such that $f = Bg$, where $g = (g_0, g_1, \dots, g_r)$. For each $n \geq N$, there is a function p_n in K_n with $\|f - p_n\|_\Delta < \sigma^n$ and there exist polynomials $p_{n0}, p_{n1}, \dots, p_{nr}$ such that $p_n = B(p_{n0}, p_{n1}, \dots, p_{nr})$. By the Lemma, there is a finite constant M such that

$$\|g_j - p_{nj}\|_I \leq M \|f - p_n\|_\Delta < M\sigma^n \quad (n \geq N; j = 0, 1, \dots, r).$$

Thus

$$\limsup_{n \rightarrow \infty} \sqrt[n]{e(g_j, P_n, I)} \leq \sigma < 1 \quad (j = 0, 1, \dots, r)$$

and so the functions $g_0, g_1, \dots, g_r$ are analytic on I by Bernstein's theorem. Thus $f = Bg$ is analytic on Δ .

Moreover, if (5) holds for f , then

$$\limsup_{n \rightarrow \infty} \sqrt[n]{e(g_j, P_n, I)} = 0 \quad (j = 0, 1, \dots, r)$$

and so each g_j has a holomorphic extension to an entire function G_j by Bernstein's theorem. Hence

$$F(z) = \sum_{j=0}^r k_j(z) G_j(\phi(z)) \quad (z \in \Omega_\phi)$$

is a holomorphic extension of f to Ω_ϕ .

A simple connectedness argument shows there exists a real number ρ in $(0, 1)$ such that $\phi^{-1}(D_\rho)$ is a *region*, where D_ρ is defined by (1). Suppose (6) holds for f . Put $\sigma = \rho$ in the above argument; then Bernstein's theorem implies that each g_j has a holomorphic extension G_j to D_ρ . If $F(z) = \sum_j k_j(z) G_j(\phi(z))$ for z in $\phi^{-1}(D_\rho)$, then F is a holomorphic extension of f to the region $\phi^{-1}(D_\rho)$.

Conversely, if f is analytic on Δ , then there exist analytic functions $g_0, g_1, \dots, g_r$ on I such that $f = Bg$, where $g = (g_0, g_1, \dots, g_r)$. By Bernstein's theorem (2) and the Lemma, inequality (4) holds for f .

Suppose, in addition, that f has a holomorphic extension F to Ω_ϕ . Cauchy's integral formula shows that

$$(7) \quad F = F_0 + F_1 + \dots + F_r,$$

where F_0 is entire and F_j is holomorphic in $\mathbb{C}_\infty \setminus \{\lambda_j\}$ (where $\mathbb{C}_\infty$ is the Riemann sphere $\mathbb{C} \cup \{\infty\}$), for $j = 1, \dots, r$. By Bernstein's theorem (3),

$$\lim_{n \rightarrow \infty} \sqrt[n]{e(F_0, P_n, [a_0, b_r])} = 0.$$

Since $P_n \subseteq K_n$ and $\Delta \subseteq [a_0, b_r]$,

$$(8) \quad \lim_{n \rightarrow \infty} \sqrt[n]{e(F_0, K_n, \Delta)} = 0.$$

To establish the same result for F_j , we put $\zeta_j(z) = (z - \lambda_j)^{-1}$. Then ζ_j maps $\mathbb{C}_\infty \setminus \{\lambda_j\}$ conformally onto $\mathbb{C}$ and maps $[-\infty, b_{j-1}] \cup [a_j, +\infty]$ onto a compact interval I_j of the real line. Also, $F_j = G_j \circ \zeta_j$, where G_j is entire. By Bernstein's theorem (3),

$$(9) \quad \lim_{n \rightarrow \infty} \sqrt[n]{e(G_j, P_n, I_j)} = 0.$$

Let $P_n(\lambda_j) = \{p \circ \zeta_j: p \in P_n\}$ for $n = 1, 2, 3, \dots$; then $P_n(\lambda_j) \subseteq K_n$. Since

$$\|G_j - p\|_{I_j} \geq \|G_j \circ \zeta_j - p \circ \zeta_j\|_\Delta = \|F_j - p \circ \zeta_j\|_\Delta \quad \left(p \in \bigcup_{n=1}^{\infty} P_n \right),$$

equation (9) implies that

$$\lim_{n \rightarrow \infty} \sqrt[n]{e(F_j, P_n(\lambda_j), \Delta)} = 0$$

and hence that

$$(10) \quad \lim_{n \rightarrow \infty} \sqrt[n]{e(F_j, K_n, \Delta)} = 0 \quad (j = 1, \dots, r).$$

By (7), (8) and (10), equation (5) holds.

This completes the proof.

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