ON ALGEBRAS GENERATED BY INVERTIBLE OPERATORS

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ABSTRACT. Let T be an invertible operator such that the compact operators in $R(T)$ are weakly dense in it. Then $T^{-1} \in R(T)$. In particular, if T is an essentially unitary C_0 operator, $T^{-1} \in R(T)$.

Let T be an invertible operator on a separable complex Hilbert space H . If H is finite dimensional, there exists a polynomial p such that $T^{-1} = p(T)$. It is natural to consider the infinite dimensional analogue of this fact. Is T^{-1} a limit of polynomials in T in some operator topology? It is easy to see that in general the answer is negative even for the weak operator topology (example: the bilateral shift). It is obvious that for T^{-1} to be a weak limit of polynomials in T , it is necessary for every invariant subspace of T to be invariant under T^{-1} . It is not known if this is sufficient (see [1]).

Let $\text{Lat } T$ denote the lattice of closed invariant subspaces of T and $R(T)$ the weak closure of the algebra of polynomials in T . Here we consider an operator T for which the compact operators in $R(T)$ are weakly dense there. As shown by Nordgren in [2], essentially unitary C_0 contractions are such operators.

For a positive integer n , $\mathcal{K}^{(n)}$ will denote the usual direct sum of n copies of $\mathcal{K}$ and $T^{(n)}$ will denote the direct sum of n copies of T .

For an algebra R , $R^{(n)} = \{A^{(n)}: A \in R\}$. The following lemma is standard.

LEMMA 1. $B \in R(T)$ if and only if $\text{Lat } T^{(n)} \subseteq \text{Lat } B^{(n)}$ for all positive integers n .

LEMMA 2. Suppose T is invertible and $\mathfrak{M}$ is a finite dimensional invariant subspace of T , then $\mathfrak{M} \in \text{Lat } T^{-1}$.

PROOF. If $\mathfrak{M} \in \text{Lat } T$, $T\mathfrak{M} \subseteq \mathfrak{M}$. Since T is invertible, it preserves dimension. Thus $\dim(T\mathfrak{M}) = \dim \mathfrak{M}$. This implies $T\mathfrak{M} = \mathfrak{M}$.

THEOREM. Let T be an invertible operator. If the compact operators in $R(T)$ are weakly dense there, then $T^{-1} \in R(T)$.

PROOF. By Lemma 1, it suffices to show that $\text{Lat } T^{(n)} \subset \text{Lat } T^{-1(n)}$. Since $T^{(n)}$ and T satisfy the same hypotheses, the argument used for the case $n = 1$

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will be applicable to $n > 1$. Thus, it is enough to show that $\text{Lat } T \subset \text{Lat } T^{-1}$. In particular, it is enough to show that every cyclic subspace of T is invariant under T^{-1} .

Denote the component of $\rho(T)$ containing zero by $\rho_0(T)$. Then for $\lambda \in \rho_0(T)$, $\text{Lat } T^{-1} = \text{Lat}(T - \lambda)^{-1}$.

For $x \in \mathcal{H}$, and $\lambda \in C$, it is clear that

$$\bigvee_{n=0}^{\infty} T^n x = \bigvee_{n=0}^{\infty} (T - \lambda)^n x.$$

Let $\mathfrak{M} = \bigvee_{n=0}^{\infty} T^n x \in \text{Lat } T$. Let $\mathfrak{N}$ be the span of all invariant subspaces of T in $\mathfrak{M}$ which are in $\text{Lat } T^{-1}$.

If $\mathfrak{N} = \mathfrak{M}$ then $\mathfrak{M} \in \text{Lat } T^{-1}$. Suppose $\mathfrak{N} \neq \mathfrak{M}$. Let $\hat{\mathcal{H}} = \mathcal{H} \ominus \mathfrak{N}$, $\hat{\mathfrak{N}} = \mathfrak{N} \ominus \mathfrak{N}$ and for an operator A on $\mathcal{H}$, $\hat{A}$ the compression of A to $\hat{\mathcal{H}}$. Since $\mathfrak{N} \in \text{Lat } T \wedge \text{Lat } T^{-1}$, $\hat{T}$ is invertible with inverse $\hat{T}^{-1}$. Thus we may as well assume $\mathfrak{N} = \{0\}$; i.e., $\mathfrak{M}$ contains no common invariant subspace of T and T^{-1} .

Let $\{A_K\} \in R(T)$ be a net of compact operators which converges to T , and let $\lambda \in \rho_0(T)$ such that $|\lambda| \geq \delta > 0$ for some δ . Then

$$\mathfrak{M} \notin \text{Lat}(T - \lambda)^{-1}.$$

Since $T - \lambda$ is invertible, $(T - \lambda)\mathfrak{M}$ is a closed subspace of $\mathfrak{M}$ and since $\mathfrak{M} = \bigvee_{n=0}^{\infty} (T - \lambda)^n x$, $\mathfrak{M} \ominus (T - \lambda)\mathfrak{M}$ has dimension one. Thus on $\mathfrak{M} = (T - \lambda)\mathfrak{M} \oplus \mathfrak{N}_\lambda$,

$$A_K|_{\mathfrak{M}} = \begin{pmatrix} B_K & C_K \\ 0 & \lambda_K \end{pmatrix}.$$

Since $\mathfrak{N}_\lambda$ is one-dimensional,

$$(T - \lambda)|_{\mathfrak{M}} = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}.$$

Since $A_K - \lambda$ converges to $T - \lambda$, it follows that λ_K converges to λ . Thus for K sufficiently large, $\lambda_K \neq 0$. Note that $\lambda_K \in \sigma(A_K|_{\mathfrak{M}})$. Since A_K is compact, λ_K is an eigenvalue of $A_K|_{\mathfrak{M}}$ and the corresponding eigenspace $\mathcal{E}_\lambda$ is finite dimensional. Since T commutes with A_K on $\mathfrak{M}$, $\mathcal{E}_\lambda \in \text{Lat } T|_{\mathfrak{M}}$ and thus $\mathcal{E}_\lambda \in \text{Lat } T$. Since $\mathcal{E}_\lambda$ is finite dimensional, $\mathcal{E}_\lambda \in \text{Lat } T^{-1}$ by Lemma 2. This contradicts the fact that $\mathfrak{M}$ contains no common invariant subspace of T and T^{-1} . The proof is complete.

COROLLARY. *If T is an essentially unitary C_0 contraction, $T^{-1} \in R(T)$.*

REMARKS. The above theorem suggests the following two questions:

(1) Let R be a commutative algebra generated by compact operators. Is R inverse closed; i.e., for $T \in R$ invertible, is $T^{-1} \in R$? It is clear that the commutativity condition is essential. For consider the following example. Let $\mathcal{H} = l^2(-\infty; \infty)$ with the usual basis $\{e_n\}_{n=-\infty}^{\infty}$. Define F_n by

$$F_n e_j = \begin{cases} e_{j+1}, & j = n, \\ 0, & j \neq n, \end{cases}$$

Then F_n is of rank one. Let R be the weakly closed algebra generated by $\{F_n\}$. Then R contains the bilateral shift W . Also $l^2(0; \infty) \in \text{Lat } R$. Thus $W^{-1} \notin R$.

(2) Suppose T is invertible and $R(T)$ contains an injective compact operator. Is $T^{-1} \in R(T)$?

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