KÄHLER METRICS ON FIBERED MANIFOLDS¹

PAUL C. YANG

ABSTRACT. A Bochner type integral identity is used to derive a nonfibration theorem for complex manifolds with negative holomorphic bisectional curvature. As an application, a classification of fibered compact Kähler surfaces with nonpositive bisectional curvature is given.

0. Introduction. In [1], Goldberg and Kobayashi introduced the notion of holomorphic bisectional curvature in order to study compact Kähler manifolds with positive tangent bundle. In this paper we derive a Bochner-type formula for the variation of volume of a factor in a product manifold involving an integral of this curvature function. As a consequence, the following theorem is derived in §1.

THEOREM. *If $F \rightarrow M \rightarrow N$ is a holomorphic fiber bundle, where F is compact, then M does not admit a Kähler metric of negative holomorphic bisectional curvature. In fact, a Kähler metric on M with nonpositive holomorphic bisectional curvature induces a Kähler metric on N making $M \rightarrow N$ a Riemannian submersion, and the metric on M is, over each trivialization, a product metric.*

In §2 we apply this observation to compact Kähler surfaces to obtain

COROLLARY 1. *If M is a compact Kähler surface with nonpositive bisectional curvature K , and such that $K(\sigma, \sigma') < 0$ for all holomorphic planes σ, σ' at some $p \in M$, then M is an algebraic surface of general type.*

Next we study the fibered Kähler surfaces with nonpositive bisectional curvature. The result is the following classification.

COROLLARY 2. *If M is a compact fibered Kähler surface with nonpositive holomorphic bisectional curvature, then M is one of the following:*

- (1) M is flat, i.e., $M = E \times E'$, the metric product of two elliptic curves with flat metric.
- (2) M is a torus bundle over a curve C of genus greater than one. In this case, the universal cover of M is $\mathbb{C} \times D$ endowed with the product of the flat metric on C and a complete metric of nonpositive curvature on D .
- (3) M is fibered over a torus with fiber a curve of genus greater than one. In this case the universal cover is $D \times \mathbb{C}$ endowed with the product of the flat

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metric on C with a complete metric of nonpositive curvature on D .

(4) M is a quotient of $C \times D$ where C is a curve of genus greater than one. In this case, C is isometric to the fiber in M , and $C \times D$ is endowed with a product of a metric of nonpositive curvature on C with a complete metric of nonpositive curvature on D .

1. Preliminary and proof of theorem. We follow the notations of Kobayashi and Nomizu [2]. Let $z^1, \dots, z^n, z^{n+1} (= w)$ be local coordinates in a neighborhood $U \times D$, U a domain in C^n , $D =$ unit disc in C . Let

$$\Phi = (-2i) \sum_{1 \leq A, B \leq n+1} g_{A\bar{B}} dz^A \wedge d\bar{z}^B$$

be the Kähler form of a Kähler metric on $U \times D$. The volume form of a level hypersurface $M_c = \{(z, w) \in U \times D | w = c\}$ is given by $\Phi_c^n = \Phi_c \wedge \dots \wedge \Phi_c$, where

$$\Phi_c = (-2i) \sum_{1 \leq i, j \leq n} g_{i\bar{j}}(z, c) dz^i \wedge d\bar{z}^j.$$

We compute the derivative of Φ_w^n with respect to w :

$$\begin{aligned} \frac{\partial^2}{\partial w \partial \bar{w}} \Phi_w^n &= n \Phi_w^{n-1} \wedge (-2i) \left[\sum_{1 \leq i, j \leq n} \frac{\partial^2 g_{i\bar{j}}}{\partial w \partial \bar{w}} dz^i \wedge d\bar{z}^j \right] \\ &\quad + n(n-1) \Phi_w^{n-2} \wedge \frac{\partial \Phi_w}{\partial w} \wedge \frac{\partial \Phi_w}{\partial \bar{w}}. \end{aligned}$$

The components of the curvature tensor being given by

$$K_{i\bar{j}k\bar{l}} = \frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^l} - \sum g^{\bar{\alpha}\beta} \frac{\partial g_{i\bar{\alpha}}}{\partial z^k} \frac{\partial g_{j\bar{\beta}}}{\partial \bar{z}^l},$$

we may write

$$\begin{aligned} \frac{\partial^2 \Phi_w}{\partial w \partial \bar{w}} &= n \Phi_w^{n-1} \wedge (-2i) \left(\sum_{1 \leq i, j \leq n} K_{i\bar{j}n+1\overline{n+1}} dz^i \wedge d\bar{z}^j \right) \\ &\quad + n \Phi_w^{n-1} \wedge (-2i) \left[\sum_{\substack{1 \leq i, j \leq n \\ 1 \leq A, B \leq n+1}} g^{\bar{A}B} \frac{\partial g_{i\bar{A}}}{\partial w} \frac{\partial g_{j\bar{B}}}{\partial \bar{w}} dz^i \wedge d\bar{z}^j \right] \\ &\quad + n(n-1) \Phi_w^{n-2} \wedge \frac{\partial \Phi_w}{\partial w} \wedge \frac{\partial \Phi_w}{\partial \bar{w}}. \end{aligned}$$

PROOF OF THEOREM. Let $F \rightarrow M \rightarrow N$ be as above, $F \times D \subset M$ be a local trivialization of M over a disc in N . Let U be a coordinate neighborhood of F ; then we are in the setup discussed in the preliminary. The function $v: D \rightarrow \mathbf{R}^+$ given by $v(w) =$ volume of fiber over w is constant (since Kähler submanifold minimizes volume in its homology class); hence $(\partial^2 v / \partial w \partial \bar{w})(w) = 0$ for all points in D ; thus if M has nonpositive holomorphic bisectional curvature ($K_{\alpha\bar{\alpha}\beta\bar{\beta}} \geq 0$), all three integrands appearing in $\partial^2 v(w) / \partial w \partial \bar{w} =$

$\int_{M_w} \partial^2 \Phi_w^n / \partial w \partial \bar{w}$ are nonnegative. Hence, each must vanish identically. As a consequence we have

$$\partial g_{i\bar{A}} / \partial w = \partial g_{w\bar{A}} / \partial z^i = \partial g_{j\bar{B}} \partial \bar{w} = \partial g_{w\bar{B}} / \partial \bar{z}^j = 0,$$

which are precisely the desired conclusions.

2.

PROOF OF COROLLARY 1. We examine the Kodaira classification of surfaces [3]. Recalling the fact that holomorphic sectional curvature decreases on a complex submanifold of a Kähler manifold, we may deduce from the Gauss-Bonnet Theorem that M does not contain P^1 . In the Kodaira list of surfaces, with the exception of the algebraic surfaces of general type, all surfaces either contain a P^1 or have vanishing first Chern class or are elliptic bundles which are ruled out by our Theorem.

PROOF OF COROLLARY 2. Suppose $M \rightarrow C$ is a Kähler surface fibered over a curve C , with nonpositive bisectional curvature, then the Theorem implies that the fibers are totally geodesic submanifolds of M , hence, in particular, the curvature K of the fiber in the induced metric agrees with the ambient holomorphic sectional curvature; we consider two cases. First, $K \equiv 0$ in one fiber (hence $\equiv 0$ for all fibers since the fibers are isometric). This implies $\tilde{F}$ is an elliptic curve. Consider the universal cover $\pi: \tilde{C} \rightarrow C$; then $\pi^*M \rightarrow \tilde{C}$ is again a Kähler surface fibered over $\tilde{C}$ with nonnegative holomorphic bisectional curvature, hence the induced metric is a product metric. In the case $\tilde{C} = \mathbb{C}$ we can conclude by the Gauss-Bonnet Theorem that the metric on $\mathbb{C}$ must be the usual flat metric, and this is statement (1). If $\tilde{C} = D$ we have (2).

Second, $K \not\equiv 0$, hence $K < 0$ somewhere, therefore the fiber F is a curve of genus greater than one. Suppose $\tilde{C} = \mathbb{C}$ then again C is an elliptic curve with flat metric. Let $\pi: \mathbb{C} \rightarrow C$ be the covering map; then our Theorem applied to π^*M , the induced fiber bundle over $\mathbb{C}$, is a Riemannian product since $\mathbb{C}$ is simply connected. This is the case covered by statement (3).

The remaining possibility is $\tilde{C} = D$; hence we have statement (4).

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DEPARTMENT OF MATHEMATICS, RICE UNIVERSITY, HOUSTON, TEXAS 77001

Current address: School of Mathematics, The Institute for Advanced Study, Princeton, New Jersey 08540