SEPARABLE CRITERIA FOR G -DIAGRAMS OVER COMMUTATIVE RINGS

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ABSTRACT. Let S be a commutative, separable algebra over the commutative ring R and finitely generated and projective as an R -module. Suppose G is a group of ring automorphisms of S stabilizing R setwise. It is shown that for the ring of invariants S^G to be a strongly separable extension of R^G it is necessary that $R \cdot S^G$ be R -separable; and it is shown that this condition is sufficient when R and S are finitely generated algebras over an algebraically closed field and G is a linearly reductive algebraic group acting rationally on S .

Recall that S is a strongly separable R -algebra if S is a separable R -algebra and a finitely generated and projective R -module [DI]. We say that S is a Galois extension of R with finite group Γ of R -algebra automorphisms of S if it satisfies the Chase-Harrison-Rosenberg definition of a Galois extension [DI]: $S^\Gamma = R$, and $l: S \otimes_R S \rightarrow C(\Gamma, S)$ is an isomorphism, where $l(s \otimes t)(\gamma) = s\gamma(t)$ and $C(\Gamma, S)$ is all functions from Γ to S . It is important for our purposes to note that if l is an epimorphism then l is an isomorphism.

DEFINITION. Let G be a subgroup of the ring automorphisms of S such that G restricted to R is contained in the ring automorphisms of R . Then we have the following diagram of inclusions:

Such a diagram is called a G -diagram of S over R .

This paper is concerned with the following question: If we have a G -diagram of S over R with S a strongly separable (Galois) R -algebra, is S^G a strongly separable (Galois) R^G -algebra? We begin by finding necessary conditions.

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PROPOSITION 1. Suppose we have a G -diagram of S over R .

- (a) If S^G is a separable R^G -separable, then $R \cdot S^G$ is a separable R -algebra.
- (b) Assume there is a finite group Γ contained in the ring automorphisms of S with $S^\Gamma = R$ and $\Gamma|S^G$ is contained in the ring automorphisms of S^G . If S^G is a Galois extension of R^G with group $\Gamma|S^G$, then $R \cdot S^G$ is a Galois extension of R with group $\Gamma|R \cdot S^G$.

PROOF. (a) Since S^G is a separable R^G , $R \otimes_{R^G} S^G$ is a separable R -algebra. Now $\mu: R \otimes_{R^G} S^G \rightarrow R \cdot S^G$, where $\mu(r \otimes s) = rs$, is an epimorphism. Hence, $R \cdot S^G$ is a separable R -algebra.

(b) Since $\Gamma|S^G$ is contained in the ring automorphisms of S^G and $S^\Gamma = R$, then $\Gamma|R \cdot S^G$ is contained in the ring automorphisms of $R \cdot S^G$ and $(R \cdot S^G)^\Gamma = R$. Hence, we need only show that $l: (R \cdot S^G) \otimes_R (R \cdot S^G) \rightarrow C(\Gamma|R \cdot S^G, R \cdot S^G)$ is surjective. Since S^G is a Galois extension of R^G , then $R \otimes_{R^G} S^G$ is a Galois extension of R with group $\Gamma|R \otimes_{R^G} S^G$. Hence, $l': (R \otimes_{R^G} S^G) \otimes (R \otimes_{R^G} S^G) \rightarrow C(\Gamma|R \otimes_{R^G} S^G, R \otimes_{R^G} S^G)$ is an isomorphism. Observe that the range and domain of l are epimorphic images of those of l' , and it follows that l is a surjection.

For the remainder of the paper we assume that G is a linearly reductive algebraic group over the algebraically closed field k acting rationally on the finitely generated k -algebras S and R . This will be called the *Reductive Case*.

Let M be a rational G -module. Recall that $M^G = \{m \in M | g(m) = m \text{ for all } g \text{ in } G\}$, and M is G -ergodic if $M^G = (0)$. Let M_G denote the unique maximal G -ergodic submodule of M . Then $M = M^G \oplus M_G$, and the projection $P_M: M^G \oplus M_G \rightarrow M^G$ is called the Reynold's operator of M [F, 5.3, p. 156]. If M is also an R -module, we call M a *compatible* G and R^G -module if $g(rm) = g(r)g(m)$ for all r in R , m in M , g in G .

LEMMA 2. Let M be a compatible G and R^G -module. Then $P_M: M^G \oplus M_G \rightarrow M^G$ is an R^G -module homomorphism. In particular, M^G and M_G are R^G -modules.

PROOF. [F, 5.4, p. 156].

LEMMA 3. Suppose that we have a G -diagram of S over R in the Reductive Case.

- (a) If there is a finite group Γ of automorphisms of S^G with $(S^G)^\Gamma = R^G$, then S^G is a finite R^G -module.
- (b) If $R \cdot S^G$ is a finite R -module, then S^G is a finite R^G -module.
- (c) If $R \cdot S^G$ is a finite and projective R -module, then S^G is a finite and projective R^G -module.
- (d) Assume that S is a strongly separable R -algebra. If S^G is a separable R^G -algebra, then S^G is a strongly separable R^G -algebra.

PROOF. (a) Since S is a finitely generated k -algebra and G is linearly reductive, S^G is a finitely generated k -algebra [F, 5.9, p. 160]; and, hence, S^G is a finitely generated R^G -algebra. Since Γ is finite and $(S^G)^\Gamma = R^G$, the ring S^G is integral over R^G . This implies that S^G is a finite R^G -module.

(b) Since $R \cdot S^G$ is a finite R -module, we have $R \cdot S^G = \sum_{i=1}^n R x_i$ where we may assume the x_i are in S^G . Then each s in S^G can be written $s = \sum r_i x_i = \sum r'_i x_i + \sum r''_i x_i = \sum r'_i x_i$ with r_i in R , $r_i = r'_i + r''_i$, r'_i in R^G , and r''_i in R_G . The last equality follows since S is a compatible G and R^G -module, by Lemma 2, and since $S = S^G \oplus S_G$. Hence S^G is generated as an R^G -module by $\{x_1, \dots, x_n\}$.

(c) Suppose $R \cdot S^G$ is a finite and projective R -module. Since $R \cdot S^G = \sum_{i=1}^n R x_i$ with x_i in S^G , there is a dual basis $\{f_i, x_i\}$ of $R \cdot S^G$ over R . Let $\Pi': R = R^G \oplus R_G \rightarrow R^G$ and $\Pi'': R = R^G \oplus R_G \rightarrow R_G$ be projections, and let $f'_i = \Pi'_i f_i$ and $f''_i = \Pi''_i f_i$. Assume x is in S^G . Then $x = \sum f_i(x) x_i = \sum f'_i(x) x_i + \sum f''_i(x) x_i = \sum f'_i(x) x_i$ by the compatibility of the G and R -module structure of $R \cdot S^G$, Lemma 2, since $(R \cdot S^G) = S^G \oplus (R \cdot S^G)_G$. Hence, $\{f'_i, x_i\}$ is a dual basis for S^G and R^G , that is, S^G is a finite and projective R^G -module.

(d) Since S^G is a separable R^G -algebra, we apply Proposition 1(a) to get that $R \cdot S^G$ is a separable R -algebra. But $R \cdot S^G$, being a separable subextension of the strongly separable extension S , implies that $R \cdot S^G$ is a strongly separable R -algebra; in particular, $R \cdot S^G$ is a finite and projective R -module. Apply (c).

Suppose that we have a G -diagram of S over R in the Reductive Case and S is a strongly separable R -algebra. Since G is linearly reductive, $R^G \hookrightarrow R$ and $S^G \hookrightarrow S$ are split monomorphisms of R^G and S^G -modules, respectively. Since S is a strongly separable R -algebra, the inclusion $R \hookrightarrow S$ splits as an R -module. Recall that: If T is any commutative R^G -algebra, then G operates by T -algebra automorphism on $R \otimes_{R^G} T$ and the action is rational. Moreover $(R \otimes_{R^G} T)^G = T$ [F, 5.6, p. 157]. Let T be an R -algebra. Hence, if we have a G -diagram

with S a strongly separable R -algebra, then we may "tensor" to get a new G -diagram:

THEOREM 4. *Suppose that we have a G -diagram of S over R in the Reductive Case with S a strongly separable R -algebra. Then S^G is a strongly separable R^G -algebra if and only if S is a projective $R \cdot S^G$ -module.*

PROOF. If S^G is a strongly separable R^G -algebra, apply Proposition 1(a).

Conversely, assume that S is a projective $R \cdot S^G$ -module, that is, $R \cdot S^G$ is a strongly separable R -algebra. Since $R \cdot S^G$ is a strongly separable R -algebra, the nilradical of R lifts to the nilradical of $R \cdot S^G$: $\text{nilrad}(R \cdot S^G) = \text{nilrad}(R)S^G$. Nilpotents map to nilpotents via ring automorphisms, thus $\text{nilrad}(R)$ is a rational G -module. Hence, $\text{nilrad}(R) = \text{nilrad}(R^G) \oplus (\text{nilrad}(R))_G$, and $\text{nilrad}(R \cdot S^G) = \text{nilrad}(R^G)S^G \oplus (\text{nilrad}(R))_G S^G$. The last equality follows since $\text{nilrad}(R^G)S^G$ is contained in S^G and since S_G is an S^G -module, $(\text{nilrad}(R))_G S^G$ is contained in S_G . Therefore, $\text{nilrad}(S^G) = \text{nilrad}(R \cdot S^G) \cap S^G = \text{nilrad}(R^G)S^G$.

Let m be a maximal ideal in R^G and consider the G -diagram

$$\begin{array}{ccc}
 & R \cdot S^G \otimes R^G/m & \\
 & \swarrow \quad \searrow & \\
 S^G \otimes R^G/m & & R \otimes R^G/m \\
 & \swarrow \quad \searrow & \\
 & R^G/m &
 \end{array}$$

By the above $\text{nilrad}(S^G \otimes R^G/m) = 0$. Since $R \cdot S^G \otimes R^G/m$ is a finite R -module, by Lemma 3(b), $S^G \otimes R^G/m$ is a finite R^G/m -module. Since G is linearly reductive, R^G is a finitely generated k -algebra; and hence, $R^G/m \simeq k$. This means we have $S^G \otimes R^G/m$ reduced and finite over the algebraically closed field R^G/m . Thus $S^G \otimes R^G/m$ is a separable R^G/m -algebra. With this happening for each maximal ideal in R^G , S^G is a separable R^G -algebra [DI, 7.1, p. 72]. And by Lemma 3(d), S^G is a strongly separable R^G -algebra.

THEOREM 5. *Suppose that we have a G -diagram of S over R in the Reductive Case and S is a Galois extension of R with group Γ such that $\Gamma|S^G$ is contained in the ring automorphisms of S^G and $(S^G)^\Gamma = R^G$. Then S^G is a Galois extension of R^G with group $\Gamma|S^G$ if and only if $R \cdot S^G$ is a Galois extension of R with group $\Gamma|R \cdot S^G$.*

PROOF. If S^G is a Galois extension of R^G , we apply Proposition 1(b).

Conversely, assume that $R \cdot S^G$ is a Galois extension of R with group $\Gamma|R \cdot S^G$. Hence, $l: R \cdot S^G \otimes_R R \cdot S^G \rightarrow C(\Gamma|R \cdot S^G, R \cdot S^G)$ is an isomorphism. Since $(S^G)^\Gamma = R^G$, we need only show that $l': S^G \otimes_{R^G} S^G \rightarrow C(\Gamma|S^G, S^G)$ is surjective. Note that $C(\Gamma|S^G, S^G)$ is contained in $C(\Gamma|R \cdot S^G, R \cdot S^G)$. Let f be in $C(\Gamma|S^G, S^G)$. Since l is surjective, $f = l(\sum r_i s_i \otimes t_i)$ where we may assume r_i in R and s_i, t_i in S^G . Hence, for each γ in Γ , we have

$$f(\gamma) = l\left(\sum r_i s_i \otimes t_i\right)(\gamma) = \sum r_i s_i \gamma(t_i) = \sum r' s_i \gamma(t_i) + \sum r'' s_i \gamma(t_i)$$

with $r'_i + r''_i = r_i$, r'_i in R^G , r''_i in R_G . But $s_i \gamma(t_i)$ is in S^G , and by Lemma 2, $\sum r'_i s_i \gamma(t_i)$ is in S^G and $\sum r''_i s_i \gamma(t_i)$ is in S_G . But $f(\gamma)$ is in S^G . Hence, $\sum r''_i s_i \gamma(t_i) = 0$ and $f(\gamma) = \sum r'_i s_i \gamma(t_i)$. This means that

$$f = l'\left(\sum r'_i s_i \otimes_{R^G} t_i\right),$$

that is, l' is surjective.

We end the paper with an example in which S is a Galois extension of R and S^G is a finite R^G -module, but S^G is not a separable R^G -algebra.

EXAMPLE 6. Assume $\text{char}(k) \neq 2$. Let m be even and relatively prime to $\text{char}(k)$ and let l_1 and l_2 be positive integers such that ml_2 divides l_1 . Let $R = k[x_1, x_2, 1/x_1]$, $S = k[x_1, x_2, 1/x_1, x_1^{1/m}]$, $\Gamma = \{\gamma_0, \dots, \gamma_{m-1} \mid \gamma_i(x_1^{1/m}) = \xi^i x_1^{1/m}\}$ where ξ is a primitive m th root of unity, and $G = \text{GL}_1(k)$ act on S by $t(x_1) = t^{l_1} x_1$ and $t(x_2) = t^{l_2} x_2$ for t in G . Note that G is a linearly reductive algebraic group acting rationally on S and S is a Galois extension of R with group Γ . The reader can check that $S^G = \{\sum_{i \geq 0} c_i (x_2^{l_2/m l_2} / x_1^{1/m})^i\}$ and $R^G = \{\sum_{i \geq 0} c_i (x_2^{l_2/m l_2} / x_1)^i\}$. By Lemma 3(a), the ring S^G is a finite R^G -module. Let $z = x_2^{l_2/m l_2} / x_1$. Then we may think of $R^G = k[z]$ and $S^G = k[z, z^{1/m}] = R^G[w] / \langle w^m - z \rangle$. But m is a unit in S^G and z is not a unit in S^G ; hence, S^G is not a separable R^G -algebra [J]. From Theorem 4 we know in this example that S^G is not a separable R^G algebra since $R \cdot S^G$ is not a separable R -algebra:

$$R \cdot S^G = R[w] - \langle w^m - x_2^{l_2/m l_2} / x_1 \rangle.$$

And m is a unit in $R \cdot S^G$ but $x_2^{l_2/m l_2} / x_1$ is not.

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