ACCESSIBLE POINTS IN THE SPECTRUM OF A FRÉCHET ALGEBRA

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ABSTRACT. For a point in the spectrum of a uniform Fréchet algebra, several types of accessibility are defined. The relationship between accessible points and peak points is studied as is the relationship of local accessibility to global accessibility.

Introduction. Let A denote a uniform Fréchet algebra and $\text{Sp } A$ denote the spectrum of A . A condition on a point x_0 in $\text{Sp } A$ that is closely related to x_0 's being a peak point for A is that x_0 be an "accessible," or "exposed," point of $\text{Sp } A$. In this paper we will define three different types of accessibility: local accessibility, accessibility, and complete accessibility. It will be shown that, if A is a Banach algebra, then each of these types of accessibility is equivalent to x_0 's being a peak point and that these equivalences contain Rossi's local peak point theorem. For Fréchet algebras we will show that x_0 is a peak point iff x_0 is completely accessible. It was shown in [3] that local peak points need not be peak points, but we will show that local accessibility implies accessibility. This last-mentioned result is a Fréchet algebra analogue of Rossi's theorem. As a corollary, we obtain the following generalization of Rossi's theorem. If x_0 is locally accessible in a uniform algebra A , then x_0 is a peak point for A . An example will be cited to show that local accessibility need not imply complete accessibility.

Main results. A will denote a uniform Fréchet algebra. Then A is the inverse limit of a sequence $\{A_n\}$ of uniform algebras and $\text{Sp } A = \bigcup \text{Sp } A_n$. The symbol $S(\theta, r)$ will denote the sector $\{z \in \mathbb{C}: 0 \leq |z| \leq r \text{ and } -\theta/2 \leq \arg z \leq \theta/2\}$. Here we are taking the range of $\arg$ to be $(-\pi, \pi]$. By $S_0(\theta, r)$ we mean $S(\theta, r) - \{0\}$. Consider a point $x_0 \in \text{Sp } A$; we will define three types of accessibility for x_0 .

DEFINITION 1. The point x_0 is locally accessible iff there are a neighborhood U of x_0 , a function f in A , and positive numbers θ and r such that $f(x_0) = 0$ and $f(U) \cap S_0(\theta, r) = \emptyset$.

DEFINITION 2. The point x_0 is accessible iff there are a function f in A , and positive real numbers θ and r_i , $i = 1, 2, \dots$, such that $f(x_0) = 0$ and $f(\text{Sp } A_i) \cap S_0(\theta, r_i) = \emptyset$, $i = 1, 2, \dots$.

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DEFINITION 3. The point x_0 is completely accessible iff there are a function f in A , and positive real numbers θ and r such that $f(x_0) = 0$ and $f(\text{Sp } A) \cap S_0(\theta, r) = \emptyset$.

OBSERVATION 1. Since A is closed under translations and rotations, the definitions above are no more restrictive than if we assume that arbitrarily located sectors are omitted.

OBSERVATION 2. Completely accessible $\rightarrow$ accessible; accessible $\rightarrow$ locally accessible if $\text{Sp } A$ is locally compact; and accessible $\rightarrow$ completely accessible if A is a Banach algebra.

THEOREM 1. A point $x_0 \in \text{Sp } A$ is a peak point for A iff x_0 is completely accessible.

PROOF. Suppose $f \in A$ and f peaks at x_0 . Then $(f - 1)(x_0) = 0$ and $(f - 1)(\text{Sp } A) \cap S_0(\pi, 1) = \emptyset$. Therefore x_0 is completely accessible.

Assume x_0 is completely accessible. Let $f \in A$ such that $f(x_0) = 0$ and $f(\text{Sp } A) \cap S_0(\theta, r) = \emptyset$. We will construct $h \in A$ such that h peaks at x_0 . Let $r_i \in (0, r/2)$ and note that $f_i(z) = r_i/(r_i - z)$ satisfies $f_i(0) = 1$, $|f_i(z)| \leq 1/\sin(\theta/2)$ for z not in $S(\theta, r)$, and $|f_i(z)| \leq r_i/(s - r_i)$ for $z \notin \Delta_s$ and $r_i < s$. Here $\Delta_s = \{z \in C: |z| < s\}$. We now refer the reader to the proof of Theorem 11.1 on p. 52 of [2]. In order to avoid confusion, please note that we are not applying Theorem 11.1 but, rather, indicating how the construction in the proof of that theorem may be used to construct a function which peaks at x_0 . If a function h is constructed as in that proof, using $M = 1/\sin(\theta/2)$, $c = \frac{1}{2}$, $F_n = \tilde{\Delta}_{1/n}$, and $h_n(z) = r_n/(r_n - z)$ for an appropriate r_n in $(0, r)$, then the function $h = (1 - s) \sum_{j=0}^{\infty} s^j h_j$, defined on $C - S_0(\theta, r)$, satisfies $h(0) = 1$ and $|h| < 1$ on $C - S(\theta, r)$. Moreover, the partial sums $(1 - s) \sum_{j=0}^n s^j h_j$ are holomorphic on a neighborhood of $f(\text{Sp } A)$. Since these partial sums converge uniformly to h on $C - S_0(\theta, r)$ which contains $f(\text{Sp } A)$, we have $hf \in A$. Since hf peaks at x_0 , x_0 is a peak point for A .

OBSERVATION 3. A point $x_0 \in \text{Sp } A$ is locally accessible if x_0 is a local peak point. We do not know if the converse is true in general. For Banach algebras, it follows from Theorems 1 and 2.

OBSERVATION 4. Since the example in [3] shows that local peak points need not be peak points, we see on the basis of Theorem 1 and Observation 1 that points which are locally accessible need not be completely accessible. The following theorem shows, however, that local accessibility implies accessibility.

THEOREM 2. If a point x_0 in $\text{Sp } A$ is locally accessible with angle θ , then for any $\epsilon > 0$ x_0 is accessible with angle $\theta - \epsilon$.

The proof consists of constructing a function $g \in A$ such that $g(\text{Sp } A_n) \cap S_0(\theta - \epsilon, r_n) = \emptyset$. The function g will be defined as the limit of a sequence $\{g_i\}$ such that $g_i \in A_i$. This sequence is defined inductively. To define g_i , $\text{Sp } A_i$ is mapped into C^{n_i} by a function Φ_i . The Cousin problem is solved in C^{n_i} to obtain a function $\bar{g}_i$ which omits a sector. The function $\bar{g}_i$ is then multiplied by

a function of the form $\exp(z_1 \sum_1^i p_j)$, where p_j is a polynomial chosen to force the sequence $\{g_i = \bar{g}_i \exp(z_1 \sum_1^i p_j)\}$ to converge and retain the property of omitting sectors.

PROOF. Choose a sequence $\{\varepsilon_i\}$ of positive real numbers such that $\sum_1^\infty \varepsilon_i < \varepsilon/2$. We suppose that $h_i, i = 1, \dots, k$, are such that $h_i \in A, h_i(x_0) = 0$ and $h_1(U) \cap S_0(\theta, \infty) = \emptyset$ where $U = \{x \in \text{Sp } A: |h_i(x)| < 2, i = 2, \dots, k\}$. Note that there is no loss in generality in assuming $h_1(U)$ omits an unbounded sector $S_0(\theta, \infty)$. Define $\Phi_0: \text{Sp } A \rightarrow C^k$ by $\Phi_0(x) = (h_1(x), \dots, h_k(x))$. Choose an open set W in C^k such that $\log z_1$ (z_1 is the first coordinate function in C^k) has a continuous branch on $W \cap \Delta$ and $\Phi_0(\text{Sp } A) \subset W \cup \Delta$. In this paper we will always assume $0 \leq \text{Im}(\log z_1) < 2\pi$. The symbol Δ will be used to denote the unit polydisc in C^k .

In what follows, we will use $\Pi_m^n, n > m$, to denote the projection of C_n onto its first m coordinates. Choose $h_{k+1}, \dots, h_{n_1} \in A$ such that the polynomially convex hull $\hat{\sigma}_1$ of $\sigma_1 = \Phi_1(\text{Sp } A_1)$ satisfies $\Pi_k^{n_1}(\hat{\sigma}_1) \subset W \cup \Delta$. Here $\Phi_1(x) = (h_1(x), \dots, h_{n_1}(x))$. Now choose a polynomial polyhedron Q_1 such that $\hat{\sigma}_1 \subset Q_1$ and $\Pi_k^{n_1} Q_1 \subset W \cup \Delta$. Let W_1 and Δ_1 denote the inverse images in Q_1 of W and Δ with respect to the projection $\Pi_k^{n_1}$. Solve the Cousin problem on Q_1 with respect to $(1/z_1)\log z_1$ on $W_1 \cap \Delta_1$ to obtain $g_1^1 \in \text{hol } \Delta_1$ and $g_2^1 \in \text{hol } W_1$ such that $g_2^1 - g_1^1 = (1/z_1)\log z_1$ on $W_1 \cap \Delta_1$. Define a function $\bar{g}_1$ in $\text{hol } Q_1$ by

$$\bar{g}_1 = \begin{cases} z_1 \exp(z_1 g_1^1) & \text{on } \Delta_1, \\ \exp(z_1 g_2^1) & \text{on } W_1. \end{cases}$$

Define $p_1 = 0$ and $g_1 = \bar{g}_1$.

Suppose that $h_1, \dots, h_{n_l}; \Phi_1, \dots, \Phi_l; \bar{g}_1, \dots, \bar{g}_l; p_1, \dots, p_l; Q_1, \dots, Q_l; g_1, \dots, g_l$ have all been defined. Choose $h_{n_{l+1}}, \dots, h_{n_{l+1}}$ such that the polynomially convex hull $\hat{\sigma}_{l+1}$ of $\sigma_{l+1} = \Phi_{l+1}(\text{Sp } A_{l+1})$ satisfies $\Pi_k^{n_{l+1}}(\hat{\sigma}_{l+1}) \subset W \cup \Delta$. Here $\Phi_{l+1}(x) = (h_1(x), \dots, h_{n_{l+1}}(x))$. Let Q_{l+1} be a polynomial polyhedron such that $\hat{\sigma}_{l+1} \subset Q_{l+1}$ and $\Pi_k^{n_{l+1}}(Q_{l+1}) \subset W \cup \Delta$. Let W_{l+1} and Δ_{l+1} be the inverse images of W and Δ in Q_{l+1} with respect to the projection $\Pi_k^{n_{l+1}}$. Solve the Cousin problem on Q_{l+1} with respect to $(1/z_1)\log z_1$ on $W_{l+1} \cap \Delta_{l+1}$ to obtain $g_1^{l+1} \in \text{hol } \Delta_{l+1}$ and $g_2^{l+1} \in \text{hol } W_{l+1}$ such that $g_2^{l+1} - g_1^{l+1} = (1/z_1) \cdot \log z_1$ on $W_{l+1} \cap \Delta_{l+1}$. Define a function $\bar{g}_{l+1}$ in $\text{hol } Q_{l+1}$ by

$$\bar{g}_{l+1} = \begin{cases} z_1 \exp(z_1 g_1^{l+1}) & \text{on } \Delta_{l+1}, \\ \exp(z_1 g_2^{l+1}) & \text{on } W_{l+1}. \end{cases}$$

Let $q_1, \dots, q_n$ be polynomials such that $Q_l = \{z \in C^{n_l}: |q_i(z)| < 1, i = 1, \dots, m\}$ and $Q_{l+1} = \{z \in C^{n_{l+1}}: |q_i(z)| < 1, i = m+1, \dots, n\}$. In what follows, whenever a function f whose domain is in C^i is treated as a function on a subset of C^{i+j} , it will be assumed that we are referring to the function $f \circ \Pi_i^{i+j}$. This multiple use of the symbol for a function seems preferable to expressions such as $g_1^l \Pi_k^{n_{l+1}}$, which would otherwise become necessary. Define

a polynomial polyhedron Q'_{l+1} in $C^{n_{l+1}}$ by $Q'_{l+1} = \{z: |q_i(z)| < 1, i = 1, \dots, n\}$. Note that $\Phi_{l+1}(\text{Sp } A_l) \subset Q'_{l+1}$ and $\pi_{n_l}^{n_{l+1}} Q'_{l+1} \subset Q_l$.

Now consider $g_l/\bar{g}_{l+1}$ as a function on $Q'_{l+1} - \{z: z_1 = 0\}$.

$$\begin{aligned} g_l/\bar{g}_{l+1} &= \bar{g}_l \exp\left(z_1 \sum_1^l p_i\right) / \bar{g}_{l+1} \\ &= \exp\left(z_1 \sum_1^l p_i\right) \times \begin{cases} \exp(z_1(g_1^l - g_1^{l+1})) & \text{on } \Delta_{l+1} \cap Q'_{l+1}, \\ \exp(z_1(g_2^l - g_2^{l+1})) & \text{on } W_{l+1} \cap Q'_{l+1}. \end{cases} \end{aligned}$$

The equalities $g_l = \bar{g}_l \exp(z_1 \sum_1^l p_i)$ and

$$\bar{g}_l = \begin{cases} z_1 \exp(z_1 g_1^l) & \text{on } \Delta_l, \\ \exp(z_1 g_2^l) & \text{on } W_l, \end{cases}$$

are part of the induction hypothesis.

On $W_{l+1} \cap \Delta_{l+1} \cap Q'_{l+1}$ we have $(g_1^l - g_1^{l+1}) - (g_2^l - g_2^{l+1}) = (g_1^l - g_2^l) - (g_1^{l+1} - g_2^{l+1}) = 0$. Hence,

$$f_{l+1} = \begin{cases} g_1^l - g_1^{l+1} & \text{on } \Delta_{l+1} \cap Q'_{l+1}, \\ g_2^l - g_2^{l+1} & \text{on } W_{l+1} \cap Q'_{l+1} \end{cases}$$

defines a function in $\text{hol } Q'_{l+1}$. Recall that we have picked a sequence $\{\epsilon_i\}$ such that $\sum \epsilon_i < \epsilon/2$. Since Q'_{l+1} is a polynomial polyhedron and contains $\Phi_{l+1}(\text{Sp } A_l)$, we can find a polynomial p_{l+1} such that

$$\left| \bar{g}_{l+1} \exp\left(z_1 \sum_1^l p_i\right) (e^{z_1 p_{l+1}} - e^{z_1 f_{l+1}}) \right| < \epsilon_{l+1}, \quad |p_{l+1} - f_{l+1}| < \epsilon_{l+1}$$

on $\Phi_{l+1}(\text{Sp } A_l)$. Define $g_{l+1} = \bar{g}_{l+1} \exp(z_1 \sum_1^{l+1} p_i)$.

We need to establish an equality and an inequality. For this purpose, consider

$$\begin{aligned} g_l/g_{l+1} &= \bar{g}_l \exp\left(z_1 \sum_1^l p_i\right) / \bar{g}_{l+1} \exp\left(z_1 \sum_1^{l+1} p_i\right) \\ &= \bar{g}_l/\bar{g}_{l+1} \exp(z_1(p_{l+1} - f_{l+1})) \exp(z_1 f_{l+1}) \\ &= \exp(z_1(f_{l+1} - p_{l+1})). \end{aligned}$$

It follows that

$$(1) \quad \frac{g_k}{g_{k+m}} = \exp\left(z_1 \sum_{i=1}^m (f_{k+i} - p_{k+i})\right)$$

on $\Phi_{k+m}(\text{Sp } A_k) - \{z: z_1 = 0\}$. Moreover,

$$\begin{aligned}
g_l - g_{l+1} &= \bar{g}_l \exp\left(z_1 \sum_1^l p_i\right) - \bar{g}_{l+1} \exp\left(z_1 \sum_1^{l+1} p_i\right) \\
&= \bar{g}_{l+1} \exp\left(z_1 \sum_1^l p_i\right) (e^{z_1 f_{l+1}} - e^{z_1 p_{l+1}}).
\end{aligned}$$

Recall that

$$\left| \bar{g}_{l+1} \exp\left(z_1 \sum_1^l p_i\right) (e^{z_1 p_{l+1}} - e^{z_1 f_{l+1}}) \right| < \varepsilon_{l+1} \quad \text{on } \Phi_{l+1}(\text{Sp } A_l).$$

Hence,

$$(2) \quad |g_k - g_{k+m}| < \sum_{k+1}^{k+m} \varepsilon_i$$

on $\Phi_{k+m}(\text{Sp } A_k)$.

It follows from inequality (2) and $\sum \varepsilon_i < \infty$ that $\{g_i \Phi_i\}$ converges uniformly on each compact subset of $\text{Sp } A$. Define $g: \text{Sp } A \rightarrow C$ by $g = \lim g_i \Phi_i$. Note that since $g_j \Phi_j \in A_i$ for $j \geq i$, we have $g|_{\text{Sp } A_i} \in A_i$ and therefore $g \in A$. Also, note that $g(x_0) = 0$.

We now show that g omits sectors. Fix n and consider g on $\text{Sp } A_n - \{x_0\}$.

$$g_n \Phi_n / g = \lim_{j \rightarrow \infty} g_n \Phi_n / g_{n+j} \Phi_{n+j}.$$

From (1) we have

$$\frac{g_n \Phi_n}{g} = \exp\left(h_1 \sum_{i=1}^{\infty} (f_{n+i} - p_{n+i}) \circ \Phi_{n+i}\right).$$

Consider

$$g_n \Phi_n = \left(z_1 \exp\left(z_1 \left(g_1^n + \sum_1^n p_i\right)\right)\right) \Phi_n \quad \text{on } \Phi_0^{-1}(\Delta).$$

Since $h_1(x_0) = 0$, there is a neighborhood V of x_0 in $\text{Sp } A$ such that on $V \cap \text{Sp } A_n$, $|h_1| < 1$ and $|\arg \exp((z_1 (g_1^n + \sum_1^n p_i)) \Phi_n)| < \varepsilon/2$. Here we are considering the range of $\arg$ to be $(-\pi, \pi]$. Since $h_1(U) \cap S_0(\theta, \infty) = \emptyset$, we have

$$\begin{aligned}
&g_n \Phi_n(U \cap V \cap \text{Sp } A_n) \cap S_0(\theta - \varepsilon/2, \infty) \\
&= \left(z_1 \exp\left(z_1 \left(g_1^n + \sum_1^n p_i\right)\right)\right) \Phi_n(U \cap V \cap \text{Sp } A_n) \cap S_0(\theta - \varepsilon/2, \infty) = \emptyset.
\end{aligned}$$

Now, on $\text{Sp } A_n \cap V$,

$$g = g_n \circ \Phi_n \exp\left(h_1 \sum_{i=1}^{\infty} (p_{n+i} - f_{n+i})\right) \Phi_{n+i}.$$

Moreover $|(p_{n+i} - f_{n+i}) \Phi_{n+i}| < \varepsilon_{n+i}$; so $|\sum_1^{\infty} (p_{n+i} - f_{n+i}) \Phi_{n+i}| < \varepsilon/2$. Recalling that $|h_1| < 1$ on V , we have

$$\left| \arg \exp \left(h_1 \sum_{i=1}^{\infty} (p_{n+i} - f_{n+i}) \Phi_{n+i} \right) \right| < \frac{\varepsilon}{2} \quad \text{on } \operatorname{Sp} A_n \cap V.$$

Since $g_n \Phi_n(U \cap V \cap \operatorname{Sp} A_n) \cap S_0(\theta - \varepsilon/2, \infty) = \emptyset$, we have

$$g(U \cap V \cap \operatorname{Sp} A_n) \cap S_0(\theta - \varepsilon, \infty) = \emptyset.$$

Since g is zero only at x_0 , we have $0 \notin g(\operatorname{Sp} A_n - U \cap V)$. Since the last set is compact, there is a $r_n > 0$ such that $|g| > r_n$ on $\operatorname{Sp} A_n - U \cap V$. We have $g(\operatorname{Sp} A_n) \cap S_0(\theta - \varepsilon, r_n) = \emptyset$. This completes the proof of Theorem 2.

COROLLARY. *If A is a uniform algebra and x_0 is locally accessible, then x_0 is a peak point for A .*

PROOF. Theorem 2 and Observation 1 imply x_0 is completely accessible. Using Theorem 1, we conclude x_0 is a peak point for A .

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