ON THE NONEXISTENCE OF A PROJECTION FROM FUNCTIONS OF x TO FUNCTIONS OF x^n

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ABSTRACT. The subspace $\phi^* C^\infty(\mathbf{R}^1) \subset C^\infty(\mathbf{R}^1)$ of all C^∞ functions of $\phi(x) = x^n$, $n = 1, 2, 3, \dots$, is a closed subspace of $C^\infty(\mathbf{R}^1)$ by Glaeser's Composition Theorem. We prove that for $n > 2$ there does not exist a linear continuous projection π from $C^\infty(\mathbf{R}^1)$ onto $\phi^* C^\infty(\mathbf{R}^1)$.

1. Introduction. Let $\phi: U \rightarrow V$, where U and V are open subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$, $m \leq n$, respectively, be a semiproper real analytic mapping with a dense set of regular points; i.e. points $x \in U$ such that $\text{rank } d_x \phi = m$. Glaeser's Composition Theorem [2] provides the completeness of $\phi^* C^\infty(V)$ —the pullback of C^∞ functions on V .

Let $n = m = 1$, U and V neighbourhoods of zero and $\phi(x) = x^q$, $q \geq 2$. The question arises whether there exists a linear continuous projection π from $C^\infty(U)$ onto $\phi^* C^\infty(V)$. For $q = 2$ the space $\phi^* C^\infty(V)$ coincides with the set of even functions [5] and the answer is positive (for example, π is the projection along the subspace of odd functions). In this paper we show that the answer to the above question for $\phi(x) = x^q$, $q > 2$, is negative; that is, we have the following.

THEOREM 1. *For every $\phi(x) = x^q$, where $q > 2$ is an integer, the subspace $\phi^* C^\infty(V)$ has no direct complementary subspace $\mathcal{E} \subset C^\infty(V)$; i.e., there does not exist a closed subspace $\mathcal{E} \subset C^\infty(U)$ such that*

- (1) $\mathcal{E} \cap \phi^* C^\infty(V) = \{0\}$;
- (2) $\mathcal{E} + \phi^* C^\infty(V) = C^\infty(U)$.

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2. Notations. We consider two exact sequences of linear continuous mappings:

$$\begin{array}{ccccccc} (1) & 0 & \rightarrow & \mathcal{F} & \rightarrow & C^\infty(U) & \xrightarrow{T} F \rightarrow 0, \\ (2) & 0 & \rightarrow & \mathcal{F}_{\text{even}} & \rightarrow & C_{\text{even}}^\infty(U) & \xrightarrow{T} F_{\text{even}} \rightarrow 0, \end{array}$$

where T is the canonical homomorphism associating to each C^∞ function f on U its formal Taylor series Tf of infinite order at $0 \in U$. The kernel of

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$T: C^\infty(U) \rightarrow F$ is the closed subspace $\mathfrak{F}$ of all functions in $C^\infty(U)$ which are flat at 0, and the factor ring $C^\infty(U)/\mathfrak{F}$ is the linear topological space F of formal power series. The spaces $\mathfrak{F}_{\text{even}}, C^\infty_{\text{even}}(U)$ and F_{even} are the closed subspaces of all even functions in $\mathfrak{F}, C^\infty(U)$ and F respectively.

3. Our result is a consequence of the following three lemmas.

LEMMA 1. Assume $\mathfrak{E}$ is a closed subspace of $C^\infty(U)$ and is a complementary subspace to $\mathfrak{F} \subset C^\infty(U)$; i.e.,

(1) $\mathfrak{F} \cap \mathfrak{E} = \{0\}$;

(2) algebraic sum $\mathfrak{F} + \mathfrak{E}$ is a closed subspace of $C^\infty(U)$.

Then $\dim_{\mathbf{R}} \mathfrak{E} < \infty$.

LEMMA 2. Lemma 1 remains valid for spaces $C^\infty_{\text{even}}(U)$ and $\mathfrak{F}_{\text{even}}$ substituted instead of $C^\infty(U)$ and $\mathfrak{F}$ respectively.

LEMMA 3. Let $\phi(x) = x^q, q > 2$. Then for odd $q, \mathfrak{F} \subset \phi^* C^\infty(V)$ and for even $q, \mathfrak{F}_{\text{even}} \subset \phi^* C^\infty(V)$.

We prove Lemmas 1, 2 and 3 in §4.

Lemma 3 is also an immediate application of [4].

PROOF OF THEOREM 1. Suppose the contrary for odd q and let $\mathfrak{E}$ be a direct complementary subspace to $\phi^* C^\infty(V)$. Then, using Lemma 3, we obtain the following diagram of linear continuous mappings

$$(3) \quad \begin{array}{ccc} & & J \\ & \swarrow & \searrow \\ C^\infty(U) & \xrightarrow{\pi_1} & C^\infty(U)/\phi^* C^\infty(V) \\ T \downarrow & \nearrow \pi_2 & \\ F & & \end{array}$$

where π_1 is the canonical projection, $J: C^\infty(U)/\phi^* C^\infty(V) \rightarrow \mathfrak{E}$ is a linear topological isomorphism and right inverse of π_1 . The mapping π_2 is defined by $\pi_2 \circ T = \pi_1$ because $\ker T = \mathfrak{F} \subset \phi^* C^\infty(V) = \ker \pi_1$. The chain and product rules allow us to identify $\pi_2: F \rightarrow C^\infty(U)/\phi^* C^\infty(V)$ with the natural mapping $\pi_3: F \rightarrow F/\phi^* F$.

Therefore

$$\dim_{\mathbf{R}} \mathfrak{E} = \dim_{\mathbf{R}} C^\infty(U)/\phi^* C^\infty(V) = \dim_{\mathbf{R}} F/\phi^* F = \infty,$$

$\mathfrak{F} \cap \mathfrak{E} = \{0\}$ and subspaces $T\mathfrak{E}$ and $\mathfrak{F} + \mathfrak{E} = T^{-1}(T\mathfrak{E})$ of F and $C^\infty(U)$ respectively are closed. The latter contradicts Lemma 1. Hence, Theorem 1 is proved for odd q .

Suppose the statement of Theorem 1 is not true for even q and let $\mathfrak{E}$ be a direct complementary subspace to $\phi^* C^\infty(V)$. Denote $\mathfrak{E} \cap C^\infty_{\text{even}}(U)$ by $\mathfrak{E}_{\text{even}}$.

Since $\mathfrak{F}_{\text{even}} \subset \phi^* C^\infty(V) \subset C_{\text{even}}^\infty(U) \subset C^\infty(U)$, the subspace $\mathfrak{E}_{\text{even}}$ is a direct complementary subspace to $\phi^* C^\infty(V)$ in $C_{\text{even}}^\infty(U)$.

Repeating the proof for odd q , using the spaces $\mathfrak{F}_{\text{even}}, C_{\text{even}}^\infty(U), F_{\text{even}}$ and $\mathfrak{E}_{\text{even}}$ instead of $\mathfrak{F}, C^\infty(U), F$ and $\mathfrak{E}$ respectively, we prove for even $q > 2$

- (1) $\dim_{\mathbf{R}} \mathfrak{E}_{\text{even}} = \dim_{\mathbf{R}} C_{\text{even}}^\infty(U) / \phi^* C^\infty(V) = \dim_{\mathbf{R}} F_{\text{even}} / \phi^* F = \infty$;
- (2) $\mathfrak{F}_{\text{even}} \cap \mathfrak{E}_{\text{even}} = \{0\}$;
- (3) the subspaces $T\mathfrak{E}_{\text{even}}$ and $\mathfrak{F}_{\text{even}} + \mathfrak{E}_{\text{even}} = T^{-1}(T\mathfrak{E}_{\text{even}})$ of F_{even} and $C_{\text{even}}^\infty(U)$ respectively are closed.

The latter contradicts Lemma 2 ($\dim_{\mathbf{R}} \mathfrak{E}_{\text{even}} < \infty$). Hence, Theorem 1 is proved for all integers $q > 2$.

4. In this section we prove Lemmas 1, 2 and 3.

PROOF OF LEMMA 1. The mapping $T: \mathfrak{E} \rightarrow F$ is a linear continuous mapping without kernel. Also $\mathfrak{F} + \mathfrak{E}$ and $T(\mathfrak{F} + \mathfrak{E}) = T\mathfrak{E}$ are closed subspaces of $C^\infty(U)$ and F respectively. Using the open mapping theorem ([1, Chapter II, §2]) we obtain that $T: \mathfrak{E} \rightarrow F$ is a linear topological isomorphism of $\mathfrak{E}$ onto a closed subspace $T\mathfrak{E}$ of F .

Now suppose $\dim_{\mathbf{R}} \mathfrak{E} = \dim_{\mathbf{R}} T\mathfrak{E} = \infty$. Every neighbourhood W of zero in F contains a closed subspace of F of a finite codimension. Therefore, for every neighbourhood $\tilde{W}$ of zero in $T\mathfrak{E}$ there exists a line $\mathfrak{L} \subset \tilde{W}$. This property does not take place for $\mathfrak{E}$ considered in the induced topology from $C^\infty(U)$. Since $T: \mathfrak{E} \rightarrow T\mathfrak{E}$ is a linear topological isomorphism, we find that our assumption $\dim_{\mathbf{R}} \mathfrak{E} = \infty$ is wrong.

PROOF OF LEMMA 2. The proof of Lemma 2 repeats the proof of Lemma 1 after substituting the spaces $C_{\text{even}}^\infty(U), \mathfrak{F}_{\text{even}}$ and F_{even} for $C^\infty(U), \mathfrak{F}$ and F respectively.

PROOF OF LEMMA 3. The proof uses arguments similar to the proof of Theorem 1 in [3]. Let us observe that $\mathfrak{F} \subset \phi^* C^0(V)$ for odd q and $\mathfrak{F}_{\text{even}} \subset \phi^* C^0(V)$ for even q . It remains to prove that $f(x) = g(\phi(x))$ and $f \in \mathfrak{F}$ imply $g \in \mathfrak{F} \subset C^\infty(V)$.

The mapping $\phi: U \rightarrow V$ is a diffeomorphism for $x \neq 0, x \in U$. Therefore g is a C^∞ function for $y \neq 0, y \in V$. Also, using the flatness of f and

$$(4) \quad f'(x) = g'(\phi(x)) \cdot \phi'_x(x), \quad x \neq 0,$$

we obtain that $g'(y)$ has a C^0 extension $g_1(y)$ equal to zero at $y = 0$ such that $g_1(\phi(x))$ is a C^∞ function (note $g_1(\phi(x)) = g'(\phi(x))$ for $x \neq 0$). The Newton-Leibniz formula

$$\int_a^b u'(t) dt = u(b) - u(a)$$

implies now the existence of $g'(0)$ and the equality $g'(0) = \lim_{y \rightarrow 0} g'(y) = 0$. Further differentiating of

$$(5) \quad f_1(x) = g_1(\phi(x)), \quad \text{for all } x,$$

proves, using the above arguments for $g'(y)$, that $((d^k/dy^k)g)(y)$ has a C^0 extension $g_k(y)$. The function $g_k(y)$ is equal to zero at $y = 0$ and $g_k(\phi(x))$ is a C^∞ function. Therefore $((d^k/dy^k)g)(0)$ exists and the equality

$$\left(\frac{d^k}{dy^k}g\right)(0) \lim_{y \rightarrow 0} \left(\frac{d^k}{dy^k}g\right)(y) = 0$$

holds. Hence, $g \in \mathcal{F} \subset C^\infty(V)$ and $f \in \phi^* C^\infty(V)$. The lemma is proved.

REFERENCES

1. N. Dunford and J. T. Schwartz, *Linear operators. I, General theory*, Interscience, New York and London, 1958. MR 22 #8302.
2. G. Glaeser, *Fonctions composées différentiables*, Ann. of Math. (2) 77 (1963), 193–209. MR 26 #624.
3. P. Milman, *The Malgrange-Mather division theorem* (to appear).
4. J. C. Tougeron, *An extension of Whitney's spectral theorem*, Inst. Hautes Études Sci. Publ. Math. No. 40 (1971), 189–248.
5. H. Whitney, *Differentiable even functions*, Duke Math. J. 10 (1943), 159–160. MR 4, 193.

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