

## THE SOLUTION BY ITERATION OF NONLINEAR EQUATIONS IN HILBERT SPACES

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**ABSTRACT.** The weak and strong convergence of the iterates generated by  $x_{k+1} = (1 - t_k)x_k + t_kTx_k$  ( $t_k \in R$ ) to a fixed point of the mapping  $T: C \rightarrow C$  are investigated, where  $C$  is a closed convex subset of a real Hilbert space. The basic assumptions are that  $T$  has at least one fixed point in  $C$ , and that  $I - T$  is demiclosed at 0 and satisfies a certain condition of monotony. Some applications are given.

**Introduction.** Let  $H$  be a real Hilbert space,  $C$  a closed convex subset of  $H$  and  $T$  a nonlinear mapping of  $C$  into  $C$  with the nonempty fixed point set  $F$  in  $C$ . The mapping  $T$  is said to be *monotone* if  $\langle Tx - Ty, x - y \rangle \geq 0$  for all  $x, y \in C$ . According to [2] and [8], the mapping  $T$  is said to be *demiclosed at 0* in  $C$  if  $\{u_k\}$  is a sequence in  $C$  which converges weakly to  $u \in C$ , and if  $\{Tu_k\}$  converges strongly to zero, then  $Tu = 0$ . In this paper we study the convergence of the sequence of iterates generated by

$$(1) \quad x_{k+1} = (1 - t_k)x_k + t_kTx_k \quad (x_0 \in C),$$

where  $t_k \in R$ ,  $k = 0, 1, \dots$ , under the basic assumptions that  $I - T$  satisfies a particular condition of monotony and that  $I - T$  is demiclosed at 0 in  $C$ .

**The main theorems.** The mapping  $T$  will be said to satisfy condition (A) if  $F$  is nonempty and if there exists a real positive number  $\lambda$  such that

$$(2) \quad \langle x - Tx, x - \xi \rangle \geq \lambda \|x - Tx\|^2, \quad \forall x \in C, \xi \in F.$$

It is obvious that (2) is a particular condition of monotony of  $I - T$ .

**THEOREM 1.** *Let  $T: C \rightarrow C$  be a nonlinear mapping, where  $C$  is a closed convex subset of  $H$ . Suppose that  $T$  satisfies condition (A),  $I - T$  is demiclosed at 0 in  $C$  and the sequence  $\{x_k\}$  generated by (1) with  $0 < a \leq t_k \leq b < 2\lambda$  belongs to  $C$ . Then  $\{x_k\}$  converges weakly to an element of  $F$ .*

**PROOF.** From (2) we obtain

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$$\begin{aligned}
\|x_{k+1} - \xi\|^2 &= \|x_k - \xi - t_k(x_k - Tx_k)\|^2 \\
&= \|x_k - \xi\|^2 - 2t_k \langle x_k - Tx_k, x_k - \xi \rangle + t_k^2 \|x_k - Tx_k\|^2 \\
&\leq \|x_k - \xi\|^2 - t_k(2\lambda - t_k) \|x_k - Tx_k\|^2.
\end{aligned}$$

Since  $2\lambda - t_k > 0$ , it follows that  $\|x_{k+1} - \xi\| \leq \|x_k - \xi\|$  and so  $\|x_k - \xi\| \rightarrow \rho_\xi$  as  $k \rightarrow \infty$  for all  $\xi \in F$ . From  $0 < a \leq t_k \leq b < 2\lambda$  and from the above relation, it follows that

$$\|x_k - Tx_k\|^2 \leq (a(2\lambda - b))^{-1} (\|x_k - \xi\|^2 - \|x_{k+1} - \xi\|^2) \rightarrow 0 \quad (k \rightarrow \infty).$$

Since the sequence  $\{x_k\}$  is bounded, there exists a subsequence  $\{x_{k_j}\}$  of  $\{x_k\}$  which converges weakly to an  $x^*$ ; since  $\{x_{k_j}\} \subset C$  and  $C$  is closed and convex (hence weakly closed), it follows that  $x^* \in C$ . Moreover,  $x^*$  is a fixed point of  $T$ , since  $x_{k_j} - Tx_{k_j} \rightarrow 0$  and  $I - T$  is demiclosed at 0 (hence  $x^* - Tx^* = 0$ ).

Suppose there are two subsequences of  $\{x_k\}$ , say  $\{u_k\}$  and  $\{v_k\}$ , which converge weakly to  $u$  and  $v$ , respectively. As above, we have that  $u$  and  $v$  are in  $F$  and, hence,

$$(3) \quad \|x_k - u\| \rightarrow \rho_u, \quad \|x_k - v\| \rightarrow \rho_v.$$

Now, consider the sequence

$$E_k = \|u_k - u\|^2 - \|v_k - u\|^2 - \|u_k - v\|^2 + \|v_k - v\|^2.$$

Since relations (3) hold for any subsequence of  $\{x_k\}$  (in particular, for  $\{u_k\}$  and  $\{v_k\}$ ), it follows that  $E_k \rightarrow 0$  as  $k \rightarrow \infty$ . On the other hand, by a simple computation, we have

$$(4) \quad E_k = -2 \langle u_k - v_k, u - v \rangle.$$

This and the weak convergence of  $\{u_k\}$  and  $\{v_k\}$  to  $u$  and  $v$ , respectively, imply that  $E_k \rightarrow -2\|u - v\|^2$  and, hence,  $\|u - v\| = 0$ , i.e.,  $u = v$ . Therefore, all weakly convergent subsequences of  $\{x_k\}$  have the same weak limit, say  $x^*$ . It follows that  $x_k \rightarrow x^*$  as  $k \rightarrow \infty$  and the theorem is proved.

**REMARK.** If  $T$  is quasi-nonexpansive (i.e.  $\|Tx - \xi\| \leq \|x - \xi\|$  for all  $x \in C$  and  $\xi \in F$ ) then (2) is satisfied with  $\lambda = \frac{1}{2}$ . In this case we obtain a result of W. G. Dotson [4], which is, in turn, a generalization of a theorem of H. Schaefer [9].

A similar condition of monotony was considered by J. B. Diaz and F. T. Metcalf [3], namely (with our notation): there exists  $\lambda > 0$  such that  $\langle x - Tx, x - \xi \rangle > (\lambda/2)\|x - Tx\|^2$  for all  $x \in C$  and  $\xi \in F$ . The above-mentioned authors have proved that if  $F$  is nonempty and  $T$  is a continuous mapping, then the sequence  $\{x_k\}$  given by (1) with  $t_k = \lambda$  either contains no strongly convergent subsequence or  $\{x_k\}$  is strongly convergent to an element of  $F$ . A similar result can be obtained under the weaker assumption that  $I - T$  is demiclosed at 0 ( $T$  is not necessarily continuous).

From the point of view of applications it is interesting to obtain additional conditions such that the sequence  $\{x_k\}$  converges strongly to an element of  $F$ .

In a recent paper, [10] H. F. Senter and W. G. Dotson considered the following condition: There is a nondecreasing function  $f: [0, \infty) \rightarrow [0, \infty)$  with  $f(0) = 0$  and  $f(r) > 0$  for  $r > 0$ , such that  $\|x - Tx\| \geq f(d(x, F))$  for all  $x \in C$ , where  $d(x, F) = \inf\{\|x - z\|: z \in F\}$ . In [7], C. Outlaw considered the more restrictive case,  $\|x - Tx\| \geq c \cdot \sup\{\|x - z\|: z \in F\}$ .

**THEOREM 2.** *Let  $T$  be as in Theorem 1. If, in addition, there is  $h \in C$ ,  $h \neq 0$ , such that  $\langle x - Tx, h \rangle \leq 0$  for all  $x \in C$ , then the sequence  $\{x_k\}$  generated by (1) with  $0 < a \leq t_k \leq b < 2\lambda$  and for suitable  $x_0$  in  $C$ , converges strongly to an element of  $F$ .*

**PROOF.** By Theorem 1, it follows that  $x_k \rightarrow x^* \in F$  as  $k \rightarrow \infty$ . Suppose that  $\langle x_0, h \rangle > \langle x^*, h \rangle$ ; then there exists  $\varepsilon > 0$  such that

$$\langle x_0 - x^*, h \rangle \geq \varepsilon \|x_0 - x^*\|^2.$$

If we suppose that

$$(5) \quad \langle x_k - x^*, h \rangle \geq \varepsilon \|x_k - x^*\|^2,$$

then from (2) and from the fact that  $\langle x - Tx, h \rangle \leq 0$ , it follows that  $\langle x_{k+1} - x^*, h \rangle \geq \varepsilon \|x_{k+1} - x^*\|^2$ , that is, (5) holds for every  $k$ . Since  $x_k \rightarrow x^*$  as  $k \rightarrow \infty$ , it follows that  $\|x_k - x^*\| \rightarrow 0$ , which proves the theorem.

**Linear equations.** As an application of Theorem 2, we obtain the convergence of some iteration scheme for linear equations in Hilbert spaces, without the assumption of compactness of the mapping. We have

**THEOREM 3.** *Let  $B: H \rightarrow H$  be a continuous linear mapping and  $f \in B(H)$ . Suppose zero is an eigenvalue of the mapping  $B$  and the following condition is satisfied.*

$$(6) \quad \langle By, y \rangle \geq \lambda \|By\|^2, \quad \forall y \in H,$$

where  $\lambda > 0$ . Then the sequence  $\{x_k\}$ , generated by  $x_{k+1} = x_k - t_k(Bx_k - f)$  with  $0 < a \leq t_k \leq b < 2\lambda$  and for suitable  $x_0$  in  $H$ , converges strongly to a solution of the equation  $Bx - f = 0$ .

**PROOF.** Apply Theorem 2 with  $C = H$  and  $Tx = x - Bx + f$ . Since  $f \in B(H)$ , there exists  $\xi \in H$  such that  $B\xi - f = 0$  and, hence,  $x - Tx = B(x - \xi)$ . It is easy to see that condition (A) is satisfied (indeed,  $F \neq \emptyset$  and if in (6) we put  $y = x - \xi$  we obtain (2)). Suppose  $x_k \rightarrow x^*$  and  $Bx_k - f \rightarrow 0$  as  $k \rightarrow \infty$ ; then  $\|Bx^* - f\|^2 = \lim \langle Bx_k - f, Bx_k - f \rangle = 0$  and, hence,  $Bx^* - f = 0$ . Therefore,  $I - T$  is demiclosed at 0. Finally, since zero is an eigenvalue of  $B$  (hence also of the adjoint  $B^*$  of  $B$ ), it follows that there exists  $h \neq 0$  such that  $B^*h = 0$ . Therefore,  $\langle B(x - \xi), h \rangle = \langle x - \xi, B^*h \rangle = 0$  for all  $x \in H$  and the theorem is proved.

**REMARK.** J. B. Diaz and F. T. Metcalf [3] proved that if  $B$  is compact, semipositive (i.e.,  $\langle Bx, x \rangle \geq 0$  for all  $x \in H$ ) and selfadjoint, then (6) is satisfied with  $\lambda = 1/\lambda_1$ , where  $\lambda_1$  is the largest eigenvalue of  $B$ .

**The relaxation method for linear inequalities.** This method, given by S. Agmon [1], T. S. Motzkin and I. I. Schoenberg [6], is a nontrivial example for Theorem 1 in the finite dimensional case.

Let  $E_n$  be the real  $n$ -dimensional Euclidean space and let  $M_i \subset E_n$  ( $i = 1, \dots, m$ ) be a family of closed convex subsets of  $E_n$  with nonempty intersection,  $\cap M_i \neq \emptyset$ . We are interested in determining an element of  $\cap M_i$ .

Let  $x \in E_n$  and  $\pi(x, i)$  be the projection of  $x$  onto  $M_i$  (if  $x \in M_i$ , then  $\pi(x, i) = x$ ). Let  $i_x$  be the least index such that

$$\|x - \pi(x, i_x)\| = \max_i \|x - \pi(x, i)\|.$$

We define the mapping  $T: E_n \rightarrow E_n$  by  $Tx = \pi(x, i_x)$ . It is clear that  $x \in \cap M_i$  if and only if  $Tx = x$ , hence if and only if  $x$  is a fixed point of  $T$ .

**THEOREM 4.** *The sequence  $\{x_k\}$  generated by*

$$(7) \quad x_{k+1} = (1 - t_k)x_k + t_k\pi(x_k, i_{x_k}) \quad (x_0 \in E_n),$$

*with  $0 < a \leq t_k \leq b < 2$ , converges to an element of  $\cap M_i$ .*

**PROOF.** Apply Theorem 1 with  $C = E_n$  and  $Tx = \pi(x, i_x)$ . From the above remark,  $F = \cap M_i \neq \emptyset$ . Let  $x \in E_n$  and  $\xi \in F$ . Since  $\pi(x, i_x)$  is the projection of  $x$  onto  $M_{i_x}$  and  $\xi \in M_{i_x}$ , it follows that  $\langle x - \pi(x, i_x), \pi(x, i_x) - \xi \rangle \geq 0$ . Therefore we have

$$\begin{aligned} \langle x - Tx, x - \xi \rangle &= \langle x - \pi(x, i_x), x - \xi \rangle \\ &= \langle x - \pi(x, i_x), x - \pi(x, i_x) \rangle + \langle x - \pi(x, i_x), \pi(x, i_x) - \xi \rangle \\ &\geq \|x - \pi(x, i_x)\|^2 = \|x - Tx\|^2, \end{aligned}$$

hence we obtain relation (2) with  $\lambda = 1$ .

It remains to show that  $I - T$  is demiclosed at 0. Let  $\{x_k\} \subset E_n$  be such that  $x_k \rightarrow x^*$  and  $x_k - Tx_k \rightarrow 0$  as  $k \rightarrow \infty$ . For each  $i$  ( $1 \leq i \leq m$ ) we have

$$\|x_k - \pi(x_k, i)\| \leq \|x_k - \pi(x_k, i_{x_k})\| = \|x_k - Tx_k\| \rightarrow 0, \quad k \rightarrow \infty.$$

Since  $\pi(x, i)$  is a continuous function for each  $i$  and  $E_n$  is a finite dimensional space (hence  $\{x_k\}$  converges strongly to  $x^*$ ), it follows that

$$\lim_{k \rightarrow \infty} \|x_k - \pi(x_k, i)\| = \|x^* - \pi(x^*, i)\| = 0,$$

for each  $i$  ( $1 \leq i \leq m$ ). Therefore,  $x^* - Tx^* = x^* - \pi(x^*, i_{x^*}) = 0$  and so  $I - T$  is demiclosed at 0.

From Theorem 1, the sequence  $\{x_k\}$  generated by (7) converges weakly (and thus strongly) to an element of  $F = \cap M_i$ . This proves Theorem 4.

**Remarks.** It is easy to see that  $Tx = \pi(x, i_x)$  is a discontinuous function in those points  $x \in E_n$  where  $\max_i \|x - \pi(x, i)\|$  is touched for more than one value of index  $i$ .

Theorem 4 is due to I. I. Eremin [5]. Our method of proof seems to be simpler.

Theorem 4 contains, as a special case, a result obtained by S. Agmon [1], T. S. Motzkin and I. I. Schoenberg [6] for the case when  $M_i$  is defined by the inequalities of the form

$$\sum_{j=1}^n a_{ij}x_j + b_i \geq 0, \quad i = 1, \dots, m.$$

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