MEASURABILITY OF PARTIAL DERIVATIVES

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ABSTRACT. Let f be a real function defined in R_n . In this note we give a sufficient condition in order that the set of points where the partial derivative $\partial f/\partial x_i$ exists is Lebesgue measurable and $\partial f/\partial x_i$ is a measurable function on this set. This result unifies and extends a number of previous results.

1. According to a classical result due to Banach, a real Lebesgue measurable function f on the real line has the following property. The set of points $\Omega(f)$ where the derivative f' exists is measurable and f' is a measurable function on $\Omega(f)$. This result has an extension to functions defined in R_n . If f is a real, Lebesgue measurable function on R_n then the set of points $\Omega(f)$ where f is differentiable is measurable and the partial derivatives of f are measurable on $\Omega(f)$. This result is due to Haslam-Jones [1]. A simple proof (due to Federer) can be found in [3, p. 268]. However, the set of points where a given first-order partial derivative of f exists need not be measurable. For an example see [2, §7]. In this note we are concerned with conditions on f which ensure that the partial derivatives are measurable.

Let $x = (x_1, \dots, x_n)$ denote a generic point in R_n . Let J be an open n -dimensional interval and let f be a real measurable function in J . (In this note the term "measurable" and all related terms will refer to Lebesgue measure.) We denote by $\Omega_i(f)$ the set of points $\{x \in J: (\partial f/\partial x_i)(x) \text{ exists and is finite}\}$.

The following results are known.

I. If f is continuous on $\tau \cap J$ for almost every line τ parallel to the x_i axis then $\Omega_i(f)$ is measurable and $\partial f/\partial x_i$ is measurable on $\Omega_i(f)$.

II (Serrin [2]). If f is monotone on $\tau \cap J$ for almost every line τ parallel to the x_i axis then $\partial f/\partial x_i$ exists and is finite a.e. in J .

III (Serrin [2]). If f is of bounded variation on $\tau \cap J$ for almost every line τ parallel to the x_i axis then there exists an equivalent function h such that $\partial h/\partial x_i$ exists and is finite a.e. in J .

The last result was obtained as a corollary of II by showing that a measurable function f satisfying the condition of III is equal a.e. to the difference of two measurable functions satisfying the condition of II.

The purpose of this note is to present another sufficient condition for the measurability of partial derivatives which provides a considerable extension of the results I-III.

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2. We start with

DEFINITION 1. Let g be a real function defined on an open interval I of the real line. If $t_0 \in I$ we set

$$(1) \quad \begin{aligned} L^+g(t_0) &= \limsup_{t \rightarrow t_0+0} g(t), & L^-g(t_0) &= \limsup_{t \rightarrow t_0-0} g(t), \\ l^+g(t_0) &= \liminf_{t \rightarrow t_0+0} g(t), & l^-g(t_0) &= \liminf_{t \rightarrow t_0-0} g(t). \end{aligned}$$

We shall say that g satisfies the (SM) condition at t_0 if,

$$(2) \quad \min(L^+g(t_0), L^-g(t_0)) \leq g(t_0) \leq \max(l^+g(t_0), l^-g(t_0)).$$

Note that if g is monotone in a neighborhood of t_0 or if g is right (or left) continuous at t_0 , then g satisfies the (SM) condition at t_0 . However, the (SM) condition does not imply either of the preceding two. The following is our main result.

THEOREM 1. Let f be a real measurable function on an open n -dimensional interval J . Suppose that for almost every line τ parallel to the x_i axis the restriction $f|_{\tau \cap J}$ satisfies the (SM) condition everywhere in $\tau \cap J$. Then $\Omega_i(f)$ is measurable and $\partial f / \partial x_i$ is a measurable function on $\Omega_i(f)$.

PROOF. Let $\delta_h^i f = h^{-1}[f(x + he^i) - f(x)]$, where h is real and different from zero and e^i is the unit vector in the direction of the x_i axis.

Set

$$(3) \quad D_i^{\text{rat}} f(x) = \lim_{\substack{r \rightarrow 0 \\ r \text{ rational}}} \delta_r^i f(x),$$

for every x in J where the limit exists. Let $\Omega'_i(f)$ denote the set of points in J where $D_i^{\text{rat}} f$ exists and is finite. Clearly, $\Omega'_i(f)$ is measurable and $D_i^{\text{rat}} f$ is a measurable function on this set. Furthermore,

$$(4) \quad \Omega_i(f) \subseteq \Omega'_i(f) \quad \text{and} \quad (\partial f / \partial x_i)(x) = D_i^{\text{rat}} f(x), \quad x \in \Omega_i(f).$$

Let τ be a line parallel to the x_i axis such that $f|_{\tau \cap J}$ satisfies the (SM) condition everywhere in $\tau \cap J$. We shall show that

$$(5) \quad \Omega_i(f) \cap \tau \supseteq \Omega'_i(f) \cap \tau.$$

Let x be a point of $\Omega'_i(f) \cap \tau$ and let $\{h_m\}_1^\infty$ be a sequence converging to zero such that $h_m \neq 0$ for all m . Applying the (SM) condition at each of the points $x + h_m e^i$ ($m = 1, 2, \dots$) one can choose two sequences of rational numbers $\{r_m\}$ and $\{r'_m\}$ such that

$$(6) \quad r_m/h_m \rightarrow 1 \quad \text{and} \quad r'_m/h_m \rightarrow 1 \quad \text{as } m \rightarrow \infty,$$

and

$$(7) \quad \begin{aligned} f(x + r'_m e^i) &> f(x + h_m e^i) - h_m/m, \\ f(x + r_m e^i) &< f(x + h_m e^i) + h_m/m, \quad m = 1, 2, \dots \end{aligned}$$

From (7) we obtain

$$\frac{r_m}{h_m} \delta_{r'_m}^i f(x) - \frac{1}{m} < \delta_{h_m}^i f(x) < \frac{r'_m}{h_m} \delta_{r'_m}^i f(x) + \frac{1}{m}, \quad m = 1, 2, \dots$$

Letting m tend to infinity we get

$$(8) \quad \lim_{m \rightarrow \infty} \delta_{h_m}^i f(x) = D_i^{\text{rat}} f(x).$$

Since $\{h_m\}$ was an arbitrary sequence in $R \setminus \{0\}$, converging to zero, we deduce that $(\partial f / \partial x_i)(x)$ exists. This proves (5). In view of our assumptions, (4) and (5) imply the conclusion of the theorem.

REMARK. It is clear from the proof that the conclusion of the theorem remains valid if we merely require that the (SM) condition holds at all but a *finite* number of points on $\tau \cap J$ for almost all lines τ parallel to the x_i axis. However, if “finite” is replaced by “countable” the conclusion of the theorem may fail. This is shown by a counterexample given in [2, §7].

Clearly, results I and II follow immediately from Theorem 1. To prove III one may argue as follows. Let f be a function as in III and set

$$h(x) = \lim_{m \rightarrow \infty} f(x + m^{-1} e^i).$$

Then h is measurable in J . If τ is a line parallel to the x_i axis such that $f|_{\tau \cap J}$ is of bounded variation, then $h|_{\tau \cap J} = f|_{\tau \cap J}$ a.e. (with respect to linear Lebesgue measure) and $h|_{\tau \cap J}$ is everywhere right continuous. Hence, $h = f$ a.e. in J and $h|_{\tau \cap J}$ satisfies the (SM) condition everywhere in $\tau \cap J$ for almost all lines τ parallel to the x_i axis. Thus Theorem 1 implies III.

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