KILLING CHARACTERISITC CLASSES BY SURGERY

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ABSTRACT. Let M be an n -dimensional C^∞ manifold and let

$$c \in H^*(BO; Z_2)$$

be a characteristic class. Suppose that c as a factor annihilates all the characteristic numbers of M . We prove that if $\dim c > (n+1)/2$ then M is cobordant to a manifold which has the class c zero, in that way answering in the affirmative a question raised by C. T. C. Wall. We examine the same question for more general cobordism theories, and for Z_p characteristic classes.

0. Introduction. We begin by describing the question that we are going to investigate. We consider a certain cobordism theory which is fixed throughout the paper. Let us call $B(n)$ its classifying space and γ_n its classifying bundle of dimension n (we can think of $B(n)$ as $BO(n)$, $BSO(n)$, $B\text{Spin}(n)$, $B\text{PL}(n)$, etc.). We assume that the cohomology of $B(n)$ is finitely generated in each dimension. Let M be a compact manifold, m -dimensional and oriented in our cobordism theory. From now on we will suppress n from $B(n)$, γ_n , etc. since it is sufficient for our purposes to take n very big with respect to m . Let S be a set of characteristic classes in our cobordism theory, in other words S consists of elements in the Z_p -cohomology group of B for various prime numbers p . We will assume that all classes of S have dimension greater than one. We assume that γ is oriented with respect to all groups which appear as groups of coefficients for a class of S , i.e. Thom classes $U \in H^*(T(\gamma); Z_p)$ have been selected for the various p 's.

Question 0.1. Suppose that all normal characteristic numbers of M which include a class of S as a factor vanish. Is it true that M is cobordant to a manifold which has all the characteristic classes S zero?

THEOREM 0.2. *Some finite multiple of the cobordism class of M contains a manifold which has all of the characteristic classes of S zero.*

THEOREM 0.3. *If all of the classes of S have dimension greater than or equal to $(m+1)/2$, then M is cobordant to a manifold which has all of the characteristic classes of S zero.*

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Theorem 0.3 answers in the affirmative a question of C. T. C. Wall (see [19, p. 17]). The result was known for the case where all classes of S have dimension strictly greater than $(m + 1)/2$ (see [19, p. 17]). For related results the interested reader can look at [1]–[3], [9], [6], [15], [20], [8]. For background information concerning cobordism and characteristic classes, a good reference is R. Stong's book [14, pp. 14–39].

In [19] Question 0.1 was analyzed using surgery techniques. In this paper we study this question from a homotopy viewpoint as follows:

We form a tower of fiberings

$$B' = B_r \rightarrow B_{r-1} \rightarrow \cdots \rightarrow B_1 \rightarrow B_0 = B$$

where B_{i+1} is a fibering over B_i with fiber an Eilenberg-Mac Lane space, so that at each stage of the tower we kill one of the classes of S .

Let γ' be the pull-back of γ over B' .

Let $\alpha: S^{n+m} \rightarrow T(\gamma)$ be the map that we get from the Pontryagin-Thom construction on the oriented manifold M (for details see [14, p. 19]), and let $\{\alpha\} \in \pi_{n+m}(T(\gamma))$ be the corresponding homotopy element. Because of transversality theory we can give homotopy interpretation to Theorems 0.2, and 0.3. Theorem 0.2 says that some finite multiple of $\{\alpha\}$ lies in the image of the induced map $\pi_{n+m}(T(\gamma')) \rightarrow \pi_{n+m}(T(\gamma))$, and Theorem 0.3 says that $\{\alpha\}$ lies in the image of this map. Of course our method applies to bundle theories where transversality holds, i.e. differential manifolds, PL manifolds and topological manifolds where $m \neq 4$.

REMARK 0.4. From the previous analysis it is clear that it is enough to prove Theorems 0.2, and 0.3 only for the case where S contains just one element. So from now on we assume that $S = \{c\}$, $c \in H^d(B; \mathbb{Z}_p)$. Let $\pi: B' \rightarrow B$ be the fibration over B with fiber $K(\mathbb{Z}_p, d-1)$ which kills class c .

Let us consider the cofibration

$$(0.5) \quad T(\gamma') \rightarrow T(\gamma) \rightarrow T(\gamma)/T(\gamma').$$

Since we are in the stable range this cofibration is a fibration too. Thus we get exact sequences in homotopy, homology and cohomology. We are going to study the space $T(\gamma)/T(\gamma')$ using an idea of W. Browder (see [4]), in that way getting hold on the map $\pi_{n+m}(T(\gamma')) \rightarrow \pi_{n+m}(T(\gamma))$.

The paper is arranged as follows: In §1 we prove that the space $T(\gamma)/T(\gamma')$ is “almost” of the homotopy type of $K(\mathbb{Z}_p, d) \wedge T(\gamma)$ and also we prove Theorem 0.2. In §2 we study the space $K(\mathbb{Z}_p, d) \wedge T(\gamma)$ and we complete the proof of Theorem 0.3.

It seems to me that it is an interesting question to see what happens in the case where some classes of S have dimension less than $(m + 1)/2$.

I would like to express my gratitude to the referee whose comments simplified greatly the exposition of this paper.

1. The space $T(\gamma)/T(\gamma')$.

DEFINITION. We set $A = T(\gamma)/T(\gamma')$.

The main technical results of this section are the following:

Under the assumptions of §0, we have

PROPOSITION 1.1. *Let us call $\delta: T(\gamma) \rightarrow A$ the natural collapsing map. Then $\delta_*(\{\alpha\}) \in \pi_{n+m}(A)$ belongs to the kernel of the Hurewicz homomorphism of the space A .*

PROPOSITION 1.2. *There is a map $g: A \rightarrow K(Z_p, d) \wedge T(\gamma)$ such that:*

- (a) *the map g^* is an isomorphism in cohomology with $Z_{p'}$ coefficients, if $p' \neq p$;*
- (b) *the map g^* is an isomorphism in Z_p -cohomology up to dimension $n + 2d - 1$; and*
- (c) *in dimension $n + 2d$, the kernel of g^* in Z_p coefficients is generated by $(i \cdot c - i^2)U$ (i is the fundamental class of $K(Z_p, d)$).*

PROOF. Essentially this is Theorem 6.1 of [4].

PROOF OF THEOREM 0.2. It is well known that, since we are in the stable range, the rational Hurewicz homomorphism $\pi_*(A) \otimes Q \rightarrow H_*(A; Q)$ is a monomorphism. (Note that A is $(n - 1)$ -connected.) By Proposition 1.1, $\delta_*(\{\alpha\}) \in \pi_*(A)$ has finite order. But from the homotopy exact sequence of fibration (0.5), it follows that some finite multiple of $\{\alpha\}$ lies in the image of the map $\pi_*(T(\gamma')) \rightarrow \pi_*(T(\gamma))$, and that ends the proof.

In order to prove Proposition 1.1, we need some preparation.

LEMMA 1.3. *Consider the map $\pi: B' \rightarrow B$.*

- (a) *π^* is an isomorphism in Z_p -cohomology if $p' \neq p$; and*
- (b) *the kernel of π^* in Z_p -cohomology is the ideal of the algebra $H^*(B; Z_p)$ generated by all elements of the form $a \cdot c$, for $a \in A_p$ (A_p is the mod- p Steenrod Algebra).*

PROOF. It follows by standard (Serre) spectral sequence arguments on the fibration $K(Z_p, d - 1) \rightarrow B' \rightarrow B$. The main ingredient of the proof is the knowledge of the cohomology of $K(Z_p, d - 1)$ (see [5]) and the knowledge of the transgressions of the elements of $H^*(K(Z_p, d - 1); Z_p)$ in the Serre spectral sequence of the fibration π (see [10, p. 104]).

LEMMA 1.4. *If $a \in A_p$, then $a \cdot c$ annihilates all of the characteristic numbers of M where it appears as a factor.*

PROOF. Let $\nu: M \rightarrow B$ be the map which is determined by the orientation of M in B -cobordism. There is a right action of A_p on $H^*(B; Z_p)$ such that

$$\langle \nu^*(xa) \cdot \nu^*(y), \{M\} \rangle = \langle \nu^*(x) \cdot \nu^*(ay), \{M\} \rangle$$

$$\text{for } a \in A_p \text{ and } x, y \in H^*(B; Z_p).$$

(For the details see [22, p. 46].) Now the proof of our lemma follows easily.

PROOF OF PROPOSITION 1.1. It is enough to prove that the composition $\delta\alpha: S^{n+m} \rightarrow T(\gamma) \rightarrow A$ induces the zero map in Z_p -cohomology for any prime p' . But this follows readily from Lemmas 1.3, 1.4, and the fact that the cofibra-

tion (0.5) induces exact sequences in cohomology.

2. The space $K(Z_p, d) \wedge T(\gamma)$.

DEFINITION. We set $F = K(Z_p, d) \wedge T(\gamma)$.

In this section we will study the space F , and we will prove that it is "almost" a product of $K(Z_{p,*})$'s up to homotopy. This combined with Proposition 1.2 will enable us to get a hold on the homotopy of A .

If p is an odd prime we define $L = K(Z_p, n + 2d) \times X_x K(Z_p, d + \dim x)$. If $p = 2$ we define $L = X_x K(Z_p, d + \dim x)$. In both cases x ranges over a Z_p -basis of $H^*(T(\gamma); Z_p)$.

We define a map $h: F \rightarrow L$, such that the pull-back of the fundamental class of $K(Z_p, n + 2d)$ is $i^2 U$ (for the case where p is an odd prime) and the pull-back of the fundamental class of $K(Z_p, d + \dim x)$ is $i \cdot x$ (recall that i is the fundamental class of $K(Z_p, d)$).

PROPOSITION 2.1. *The map h has the following properties:*

- (a) h^* is an isomorphism in $Z_{p'}$ -cohomology if $p' \neq p$, and
- (b) h^* is an isomorphism in Z_p -cohomology up to dimension $n + 2d$.

PROOF. Essentially this is Theorem 6.2 of [4].

DEFINITION 2.2. Let $E = X_x K(Z_p, \dim x)$, where x runs over a basis of the cokernel of g^* (in Z_p -cohomology). Let $g': A \rightarrow E$ be the map which pulls-back the fundamental class of $K(Z_p, \dim x)$ to x . Let v be the d -dimensional Wu class of γ which is defined by the equality $v \cdot U = \chi(\text{Sq}^d)U$. It is well known (actually it follows easily from Proposition 2.3 of [4]) that v as a normal characteristic class is zero for all m -dimensional manifolds where $m < 2d$. So, as far as Theorem 0.3 is concerned, we lose nothing by assuming that $v \neq c$. Because of this assumption we can choose a Z_2 basis for $H^*(T(\gamma); Z_2)$ which includes $(v - c)U$. Furthermore we can assume that such a basis had been used in the definition of L (see definition above). Then we define L' as follows:

- (a) If $p = 2$, we define L' by splitting-off L the factor $K(Z_2, n + d)$ which corresponds to the element $(v - c)U$.
- (b) If $p \neq 2$, we define L' by splitting-off L the factor $K(Z_p, n + 2d)$ which corresponds to the element $i^2 U$.

Let $s: L \rightarrow L'$ be the natural projection. Consider the composition of maps:

$$(2.3) \quad A \xrightarrow{g \times g'} F \times E \xrightarrow{h \times \text{Id}} L \times E \xrightarrow{s \times \text{Id}} L' \times E$$

and let us call H this composition.

PROPOSITION 2.4. *The map H^* is an isomorphism in Z_p -cohomology up to dimension $(n + 2d)$.*

PROOF. We will do the case $p = 2$, the case of an odd prime being simpler.

Let us consider the element $(i \cdot c - i^2)U \in H^{n+2d}(F; Z_2)$, which, according to Proposition 1.1, belongs to the kernel of g^* . This element can be written

$$\begin{aligned}(i \cdot c - i^2)U &= (i \cdot c - \text{Sq}^d i)U \\ &= (i \cdot c)U - i \cdot \chi(\text{Sq}^d)U - \sum_1^d \text{Sq}^j(i \cdot \chi(\text{Sq}^{d-j})U)\end{aligned}$$

(for the justification of the above step see [23])

$$= i \cdot (c - v)U - \sum_1^d \text{Sq}^j(\dots).$$

From this it is clear that $i \cdot (v - c)U$ belongs to the image of H^* .

Now we are in a position to prove that H^* is onto up to dimension $n + 2d$. Really $(g \times g)^*$ and h^* are onto in this range, and the only problem could arise from the map s , but the previous argument covers this case. The map H^* is a monomorphism too in the required range, since by splitting-off a factor from L , we eliminated the kernel of g^* . The case where p is an odd prime is even simpler to prove.

PROOF OF THEOREM 0.3. Since $L' \times E$ is a product of $K(Z_p^*)$'s, by the previous proposition, it follows that the Hurewicz homomorphism of the space A is a monomorphism up to dimension $n + 2d - 1$, and since $m < 2d$, the result follows because of Proposition 1.1.

BIBLIOGRAPHY

1. D. W. Anderson, E. H. Brown, Jr. and F. P. Peterson, *The structure of the spin cobordism ring*, Ann. of Math. (2) **86** (1967), 271–298. MR **36** #2160.
2. ———, *SU-cobordism, KO-characteristic numbers, and the Kervaire invariant*, Ann. of Math. (2) **83** (1966), 54–67. MR **32** #6470.
3. ———, *Spin cobordism and related topics*, Comment. Math. Helv. **44** (1969), 462–468. MR **41** #6226.
4. William Browder, *The Kervaire invariant of framed manifolds and its generalizations*, Ann. of Math. (2) **90** (1969), 157–186. MR **40** #4963.
5. Henri Cartan, *Sur les groupes d'Eilenberg-Mac Lane*. II, Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 704–707. MR **16**, 390.
6. P. E. Conner and E. E. Floyd, *Torsion in SU-cobordism*, Mem. Amer. Math. Soc. No. 60 (1966). MR **32** #6471.
7. A. Dold, *Vollständigkeit der Wuschen Relationen zwischen den Stiefel-Whitneyschen Zahlen differenzierbarer Mannigfaltigkeiten*, Math. Z. **65** (1956), 200–206. MR **18**, 143.
8. R. Lashof, *Poincaré duality and cobordism*, Trans. Amer. Math. Soc. **109** (1963), 257–277. MR **27** #6281.
9. J. Milnor, *On the Stiefel-Whitney numbers of complex manifolds and spin manifolds*, Topology **3** (1965), 223–230. MR **31** #5207.
10. R. Mosher and M. Tangora, *Cohomology operations and applications to homotopy theory*, Harper and Row, New York and London, 1968. MR **37** #2223.
11. J. Reed, *Killing cohomology classes by surgery*, Proc. Advanced Study Inst. on Algebraic Topology (August, 1970), Vol. II, Mat. Inst., Aarhus Univ., Aarhus, 1970, pp. 446–454. MR **42** #6846.
12. J. P. Serre, *Homologie singulière des espaces fibres*, Ann. of Math. (2) **54** (1951), 425–504. MR **13**, 574.
13. ———, *Groupes d'homotopie et classes de groupes abéliens*, Ann. of Math. (2) **58** (1953), 258–294. MR **15**, 548.
14. R. E. Stong, *Notes on cobordism theory*, Math. Notes, Princeton Univ. Press, Princeton, N. J., 1968. MR **40** #2108.

15. ———, *Cobordism and Stiefel-Whitney classes*, *Topology* **4** (1965), 241–256. MR **34** #5095.
16. E. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966. MR **35** #1007.
17. N. E. Steenrod and D. B. A. Epstein, *Cohomology operations*, Ann. of Math. Studies, no. 50, Princeton Univ. Press, Princeton, N. J., 1962. MR **26** #3056.
18. R. Thom, *Quelques propriétés globales des variétés différentiables*, *Comment. Math. Helv.* **28** (1954), 17–86. MR **15**, 890.
19. C. T. C. Wall, *Surgery on compact manifolds*, Academic Press, New York, 1970.
20. ———, *Determination of the cobordism ring*, *Ann. of Math. (2)* **72** (1960), 292–311. MR **22** #11403.
21. A. Wallace, *Modifications and cobounding manifolds*, *Canad. J. Math.* **12** (1960), 503–528. MR **23** #A2887.
22. E. Brown and F. Peterson, *Relations among characteristic classes. I*, *Topology* **3** (1964), 39–52. MR **29** #629.
23. S. Papastavridis, *A formula for the obstruction to transversality*, *Topology* **11** (1972), 415–416. MR **47** #1054.

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