ON THE SUPPORT OF CERTAIN SYMMETRIC STABLE PROBABILITY MEASURES ON TVS

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ABSTRACT. Let E be a LCTVS, and let μ be a τ -regular symmetric stable probability measure of type $\alpha \in [1, 2]$ on E . Then, it is shown that the topological support S_μ of μ is a (closed) subspace of E . Further, if μ satisfies an additional regularity condition, then it is shown that S_μ is the E -closure of a Banach space contained in E . Specializing these results for $\alpha = 2$, simple proofs of all the known results regarding the support of centered Gaussian probability measures on LCTV spaces are obtained as corollaries.

1. Introduction. Recently, Kuelbs [12], Kallianpur [9] and Jain and Kallianpur [8] have shown that the topological support (support) S_μ of a centered Gaussian probability (prob.) measure μ on a real separable Banach space B is a closed subspace of B and that it is the B -closure of a separable Hilbert space H contained in B . Later an alternative proof of this was also given by Le Page [14]. The authors of [8], [9] and [14] have also shown that H is congruent to the reproducing kernel Hilbert space (RKHS) of μ . More recently, Rajput [17], Kallianpur and Nadkarni [10] and Kuelbs [13] proved analogs of this result for centered Gaussian prob. measures defined, respectively, on separable Fréchet spaces, weak duals of separable Fréchet spaces and strict inductive limits of separable Fréchet spaces. In [17], it was shown that the support of a K -regular centered Gaussian prob. measure on a complete separable locally convex topological vector space (TVS) E is a closed subspace of E . The question whether this last result holds in an arbitrary locally convex TVS was investigated by Kuelbs in [13] and, finally, settled in the affirmative by Badrikian and Chevet in [2], [3].

The purpose of this paper is to prove results similar to those mentioned above regarding the support of certain symmetric stable prob. measures on real LCTV spaces. Specifically, we show that if E is a LCTVS and if μ is a τ -regular symmetric stable prob. measure of type $\alpha \in [1, 2]$ on E , then S_μ , the support of μ , is a (closed) subspace of E (Theorem 3.1). Further, if μ satisfies an additional regularity condition, then we show that S_μ is the E -closure of a suitably chosen Banach space B contained in E (Theorem 3.2). By specializing

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these results for $\alpha = 2$, we obtain simple proofs of all the known results (indeed slight extensions thereof) regarding the support of centered Gaussian prob. measures on TV spaces mentioned in the previous paragraph (Corollaries 3.1, 3.2).

2. Preliminaries. Unless stated otherwise, the following notation and conventions will remain fixed throughout the paper. We will also list a few definitions, which will be needed in the sequel.

If X is a topological space, then $\mathfrak{B}_X$ (or just $\mathfrak{B}$, when X is known from the context) will denote the σ -algebra generated by the class of open sets of X . E and E' will denote, respectively, a real infinite dimensional Hausdorff TVS and its topological dual. By a prob. measure on E we will always mean that it is defined on $\mathfrak{B}_E$. If μ is a prob. measure on E , then $\hat{\mu}$ will denote its characteristic function, and $F_{E',\mu}$ and $G_{E,\mu}$ will denote, respectively, the sets

$$(2.1) \quad \{y \in E': \langle \cdot, y \rangle = 0, \text{ a.s. } [\mu]\}$$

and

$$(2.2) \quad \bigcap_{y \in F_{E',\mu}} N_y,$$

where $\langle \cdot, \cdot \rangle$ and N_y denote, respectively, the natural duality between E and E' and the null space of y . When E is known from context, we will write F_μ and G_μ , respectively, for $F_{E',\mu}$ and $G_{E,\mu}$.

If $a \in R$, the set of reals, $a \neq 0$ and μ a prob. measure on E , then $T_a\mu$ will denote the measure defined by $T_a\mu(A) = \mu(a^{-1}A)$, for every $A \in \mathfrak{B}_E$. If μ and ν are two τ -regular prob. measures on E , then μ^{*n} and $\mu * \nu$ will denote, respectively, μ convoluted with itself n times and the convolution of μ and ν (for the definition of τ -regular and K -regular measures and their convolutions and related concepts see [5]). If μ is a τ -regular prob. measure on a topological space then the *support* (the smallest closed set with full measure) always exists [5, p. 37]; throughout, the support of μ will be denoted by S_μ . Now we give the definition of a stable prob. measure on E .

DEFINITION 2.1. Let μ be a τ -regular prob. measure on a TVS E , then μ is called symmetric stable if μ is symmetric and for every $a > 0$, $b > 0$, there exists $c > 0$ such that

$$(2.3) \quad T_a\mu * T_b\mu = T_c\mu.$$

3. Statements and proofs of results. For the proofs of the main results we will need the following lemmas.

LEMMA 3.1 [5, p. 37]. *Let μ and ν be τ -regular prob. measures on a TVS E (E is not necessarily locally convex), then $S_{\mu*\nu} = \overline{S_\mu + S_\nu}$, where $\overline{S_\mu + S_\nu}$ is the closure of $S_\mu + S_\nu$ in E .*

LEMMA 3.2 [4]. *Let μ be a τ -regular symmetric stable prob. measure on a*

LCTVS E . Then for any positive integer n there exists $a_n > 0$ such that

$$(3.1) \quad \mu^{*n} = T_{a_n} \mu.$$

Further, the only possible constants c in (2.3) and a_n in (3.1) are respectively, $(a^\alpha + b^\alpha)^{1/\alpha}$ and $n^{1/\alpha}$, where $\alpha \in (0, 2]$ and is uniquely determined by μ (we call this α the type of μ).

LEMMA 3.3 [16, p. 187]. If ξ is a nondegenerate symmetric real r.v. of type $\alpha \in (0, 2]$, then ξ has positive density on every point of R .

LEMMA 3.4. Let Λ be a directed set and $\{\xi_\nu: \nu \in \Lambda\}$ be a net of real symmetric stable r.v.'s of type $\alpha \in (0, 2]$ on some probability space $(\Omega, \mathfrak{F}, P)$. If $\xi_\nu \rightarrow 0$ in probability, then $d_{\xi_\nu} \rightarrow 0$, where d_{ξ_ν} is the constant appearing in the characteristic function of ξ_ν , i.e., d_{ξ_ν} is given by $E(e^{it\xi_\nu}) = e^{-d_{\xi_\nu}|t|^\alpha}$, $t \in R$, (E denotes the expected value).

PROOF. Using the well-known inequality, $|e^{ir} - e^{is}| \leq \min\{2, |r - s|\}$, for $r, s \in R$, one obtains $|E(e^{it\xi_\nu} - 1)| \leq E(\min\{2, |\xi_\nu|\}) \leq \delta P\{|\xi_\nu| \leq \delta\} + 2P\{|\xi_\nu| > \delta\}$, for any $0 < \delta < 2$. This and the convergence in probability of $\{\xi_\nu\}$ to 0 yields the result.

We need a few more notation and conventions, which we list in the following:

(P-1) Let μ be a τ -regular symmetric stable prob. measure on a LCTVS E of type $\alpha \in [1, 2]$ and let $\mathcal{L}$ denote the real vector space of equivalence classes $\tilde{y}$ of μ -a.e. equal functions to $\langle \cdot, y \rangle$, for all $y \in E'$. If $\tilde{y} \in \mathcal{L}$ and $y \in \tilde{y}$, then clearly $\langle \cdot, y \rangle$ is a real symmetric stable random variable. Therefore, ϕ_y , the characteristic function of $\langle \cdot, y \rangle$ is given by $\phi_y(t) = \exp\{-d_{\tilde{y}}|t|^\alpha\}$, $t \in R$, where $d_{\tilde{y}} \geq 0$ and depends only on $\tilde{y}$ (see [15, p. 327]). From the work of Schilder [19, p. 413], we have that $\mathcal{L}$ is a normed linear space under the norm $\|\tilde{y}\|_{L_\alpha} = d_{\tilde{y}}^{1/\alpha}$. Throughout, we denote the completion of $\mathcal{L}$ in $\|\cdot\|_{L_\alpha}$ by L_α .

(P-2) Let μ be a centered Gaussian prob. measure on a LCTVS E (i.e., $\langle \cdot, y \rangle$ is a zero mean real Gaussian r.v. on $(E, \mathfrak{B}_E, \mu)$, for every $y \in E'$). Throughout, $H(\mu)$ and $H(k)$ will denote, respectively, the $L_2(E, \mathfrak{B}, \mu)$ -closure of equivalence classes $\tilde{y}$ of μ -a.e. equal functions to $\langle \cdot, y \rangle$, for all $y \in E'$, and the RKHS of k (or of μ), where k is the covariance function of μ . It is well known that $H(\mu)$ is isometrically isomorphic (congruent) to $H(k)$ under the linear extension of the map $\tilde{y} \leftrightarrow k(\cdot, y)$, $y \in E'$, and that $H(k)$ is characterized as the unique Hilbert space of real functions on E' containing the linear manifold $\{k(\cdot, y): y \in E'\}$ and satisfying $f(y) = \langle f, k(\cdot, y) \rangle_{H(k)}$, for every $y \in E'$ and $f \in H(k)$, where $\langle \cdot, \cdot \rangle_{H(k)}$ denotes the inner product in $H(k)$.

Now we are ready to state and prove our main results.

THEOREM 3.1. Let μ be a τ -regular symmetric stable prob. measure of type $\alpha \in [1, 2]$ on a LCTVS E . Then S_μ is a (closed) subspace of E . Further, $S_\mu = G_{E, \mu}$ (see (2.2) for the definition of $G_{E, \mu}$).

PROOF. First let $\alpha > 1$. Noting the simple fact $S_{T_a\mu} = aS_\mu$, for $a > 0$, it follows, from Lemmas 3.1 and 3.2, that

$$(3.2) \quad aS_\mu + bS_\mu \subseteq (a^\alpha + b^\alpha)^{1/\alpha} S_\mu,$$

for any $a > 0$ and $b > 0$; and

$$(3.3) \quad S_\mu + S_\mu + \cdots + S_\mu \subseteq n^{1/\alpha} S_\mu,$$

for any positive integer n , where there are n terms in the sum on the left side of (3.3). Clearly (3.3) implies

$$(3.4) \quad n^{1-1/\alpha} S_\mu \subseteq S_\mu.$$

Taking $a = b = 1$ in (3.2) and noting that $S_\mu = -S_\mu$ (recall that μ is symmetric) we have

$$(3.5) \quad S_\mu \pm S_\mu \subseteq 2^{1/\alpha} S_\mu.$$

(3.5) implies, in particular, that $0 \in S_\mu$. Letting $a \in (0, 1)$ and $b = (1 - a^\alpha)^{1/\alpha}$ in (3.2), one gets $aS_\mu + bS_\mu \subseteq S_\mu$. Consequently, since $0 \in S_\mu$ and $S_\mu = -S_\mu$, we have

$$(3.6) \quad aS_\mu \subseteq S_\mu$$

for any $|a| \leq 1$. Now let d be any real number; then, since $\alpha > 1$, we can choose a positive integer n_0 such that

$$(3.7) \quad n_0^{1-1/\alpha} \geq \max(|d|, 2^{1/\alpha}).$$

From (3.4), (3.6) and (3.7), we have $dS_\mu \subseteq S_\mu$; and, from (3.4)–(3.7), we have $S_\mu + S_\mu \subseteq S_\mu$. Consequently, S_μ is a subspace of E .

Now let $\alpha \in [1, 2]$. We will show that $S_\mu = G_{E,\mu}$, which will complete the proof. From the above proof, we have that S_μ is convex and circled (when $\alpha = 1$, convexity of S_μ follows from the sentence following (3.5)). Clearly $S_\mu \subseteq G_{E,\mu}$. We will now show that S_μ is indeed equal to $G_{E,\mu}$. If not, then there would exist an $x_0 \in G_{E,\mu} \setminus S_\mu$, $x_0 \neq 0$ (note $0 \in S_\mu$); and hence, since E is locally convex and S_μ is convex and closed, we can choose a $y \in E'$ such that $\langle y, x_0 \rangle$ is smaller than $r \equiv \inf_{x \in S_\mu} \langle y, x \rangle$ [11, p. 119]. Thus $S_\mu \subseteq \{x \in E: \langle y, x \rangle \geq r\}$, hence $\mu\{x \in E: \langle y, x \rangle \geq r\} = 1$. From Lemma 3.3 it follows that $\langle y, \cdot \rangle$ is degenerate at 0. It follows $N_y \supseteq G_{E,\mu}$ (see (2.2) for the definition of N_y), hence $\langle y, x_0 \rangle = 0$. A contradiction, since $\langle y, x_0 \rangle$ is strictly smaller than $r \leq 0$ (note $0 \in S_\mu$).

THEOREM 3.2. *Let μ be a τ -regular symmetric stable prob. measure of type $\alpha \in [1, 2]$ on a locally convex TVS E . Assume*

$$(3.8) \quad \sup\{\mu(C): C \text{ convex, circled, } \sigma(E, E')\text{-compact subset of } E\} = 1.$$

Then there exists a Banach space B such that $B \subseteq E$ and $S_\mu = \bar{B}$, where $-$ denotes the closure in E . Further, if $\alpha = 2$ then $B = H$, a Hilbert space congruent to the RKHS $H(k)$ of μ .

PROOF. Let L_α be as defined in (P-1) and let $\psi: E' \rightarrow L_\alpha$ be the map which takes $y \in E'$ into the corresponding equivalence class in L_α . We assert that ψ is continuous on E' , when E' is endowed with the Mackey topology M [18, p. 131]. To see this, let $\{y_\nu\}$ be a net in E' with $y_\nu \rightarrow 0$ in M -topology. Then, from (3.8) and Lemma 3.4, $\|\psi(y_\nu)\|_{L_\alpha} \rightarrow 0$; consequently, ψ is continuous at 0 and hence on E' . Since the Mackey topology is consistent with the duality $\langle \cdot \rangle$ [18, p. 131]; i.e., $(E', M)' = E$, it follows that ${}^t\psi$, the transpose of ψ , is continuous from L'_α into E , when L'_α (the dual of L_α) is endowed with the dual norm $\|\cdot\|_{L'_\alpha}$ topology and E with the strong topology (Theorem 21.6 of [11, p. 204]). Since strong topology is at least as fine as the initial topology on E , it follows that ${}^t\psi$ is also continuous when E is endowed with the initial topology. Now we show that ${}^t\psi$ is 1-1. To see this let ${}^t\psi(\xi) = 0$, $\xi \in L'_\alpha$, then $\langle {}^t\psi(\xi), y \rangle = 0$, for all $y \in E'$; therefore, $\xi(\psi(y)) = 0$, for all $y \in E'$. Thus, since $\psi(E')$ is norm-dense in L_α , $\xi = 0$. Now let $B = {}^t\psi(L'_\alpha)$; and if $x \in B$ with $x = {}^t\psi(\xi)$, define the norm of x by $\|x\| = \|\xi\|_{L'_\alpha}$. Then, clearly, B is a Banach space under the norm $\|\cdot\|$.

Now we show that $\bar{B} = S_\mu$, which, because of Theorem 3.1, is equivalent to showing that $\bar{B} = G_\mu$. First, we show that $B \subseteq N_y$, for any $y \in F_\mu$ (see (2.1) and (2.2)); this will imply B and hence $\bar{B} \subseteq G_\mu$. Let $y \in F_\mu$ be fixed and let $x \in B$; then $x = {}^t\psi(\xi)$, for some $\xi \in L'_\alpha$. Now $\langle x, y \rangle = \langle {}^t\psi(\xi), y \rangle = \xi(\psi(y))$; thus, since $\psi(y) = 0$, $\langle x, y \rangle = 0$, implying $x \in N_y$. The proof of the fact that B is indeed equal to G_μ follows now from an application of Hahn-Banach theorem.

Finally, if $\alpha = 2$, then clearly μ is centered Gaussian. Now, using the obvious fact that L_α is congruent to $H(\mu)$ which in turn is congruent to $H(k)$, one can replace L_α in the above proof, by $H(k)$. Then it is immediate that B is congruent to $H(k)$. This completes the proof.

REMARKS. (i) With a slight change in the proof of Theorem 3.2 it can be shown that when $\alpha = 2$, the space B in this theorem can be taken equal to $H(k)$. We also point out that the method of embedding of L'_α in Theorem 3.2 is similar to the one used in Theorem 4 of [7].

(ii) Lemma 3.2 holds in any TVS E in which E' separates points of E . Indeed this result is shown to be true in an arbitrary TVS E ; however, in this case α is known to belong only to $(0, \infty)$ (see [4]).

(iii) Remark (ii) and the first paragraph of the proof of Theorem 3.1 shows that if $\alpha > 1$ then the first conclusion of this theorem holds in any TVS.

(iv) A result similar to the first part of Theorem 3.1 for $1 < \alpha \leq 2$ is also obtained recently by de Acosta [6]. However, there are differences between the assumptions used in [6] and in the present paper. Though two sets of hypotheses do not seem to be directly comparable, the result of the present paper, in a sense, is more general since it holds in an arbitrary TVS (see

Remark (iii)) and does not assume—as is done in [6]—that $\mathfrak{B}_E$ coincides with the σ -algebra generated by cylinder sets. We may also note that our result was obtained independently without the knowledge of the mentioned result of [6].

Now we will show how Theorems 3.1 and 3.2 imply the known results about the support of centered Gaussian prob. measures. First a proposition whose proof is routine and is therefore omitted:

PROPOSITION 3.1. *Let E' separate points of E and μ be a prob. measure on E . Assume either (i) μ is K -regular or (ii) μ is τ -regular and $\mathfrak{B}_E$ coincides with $\mathcal{Q}$, the σ -algebra generated by $\sigma(E, E')$ -open sets. Then if μ is centered Gaussian, then μ is symmetric stable with $\alpha = 2$.*

Remark (iii), Theorems 3.1 and 3.2, and Proposition 3.1, immediately yield the following two corollaries. The first of which includes and indeed extends the result due to Kallianpur, Kuelbs, Badrikian, Chevet, Le Page, Rajput, Jain and others, mentioned in the last sentence of the first paragraph of §1. Part (i) of this corollary is new to the extent that it is proved here without local convexity hypothesis on E as opposed to the known proofs [2], [3] where this assumption seems essential. Also this part and the Hahn-Banach theorem give yet another proof of Corollary 1 of [2, p. 14].

From Remark (i), and the facts that in a quasi-complete LCTVS the closed convex hull of a compact set is compact [18, p. 189], that in a Lusin space [1] every prob. measure is K -regular and that the separable Fréchet spaces and their duals with weak or compact convergence topology as well as the strict inductive limit of sequences of separable Fréchet spaces are quasi-complete and Lusin [1, 18], Corollary 3.2 extends the results, due to Kallianpur and others, mentioned in the first three sentences of §1.

COROLLARY 3.1. *Let E be a TVS such that E' separates points of E and let μ be a centered Gaussian prob. measure on E . Assume either (i) μ is K -regular or (ii) μ is τ -regular and $\mathfrak{B}_E = \mathcal{Q}$, the σ -algebra generated by $\sigma(E, E')$ -open sets. Then S_μ is a (closed) subspace of E .*

COROLLARY 3.2. *Let μ be a τ -regular centered Gaussian prob. measure on a LCTVS E . Assume that μ satisfies (3.8) and that the characteristic functions determine measures uniquely on $\mathfrak{B}_E$ (in view of Proposition 3.1, this hypothesis can be replaced either by (i) μ is K -regular or (ii) $\mathfrak{B}_E = \mathcal{Q}$). Then $H(k) \subseteq E$, and $S_\mu = \overline{H(k)}$, where $\overline{}$ denotes the E -closure.*

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Note (added in proof on September 18, 1976). At the time this paper was in press, we were able to prove the following theorem: Let μ be a τ -regular symmetric infinitely divisible (respectively, stable of any index $\alpha \in (0, 2]$) probability measure on a LCTVS E , then S_μ is a (closed) subgroup (respective-

ly, a subspace) of E . This completes Theorem 3.2 and also provides an affirmative solution to a question raised by Kuelbs and Mandrekari in *Studia Math.* **50** (1974), 149–162. This result has been submitted for publication under the title *On the algebraic structure of the support of symmetric stable and infinitely divisible probability measures on LCTVS.*

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