ISOTOPY OF ABELIAN QUASIGROUPS

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ABSTRACT. It is proved that every abelian quasigroup possesses a class of isomorphic principal isotopes which are commutative groups. Also it is indicated how the corresponding result for topological quasigroups can be utilised to obtain results for abelian topological quasigroups from analogous results for commutative topological groups.

A quasigroup G is defined to be a system of three compositions viz. product (' $\cdot$ '), right division ('/') and left division (' $\backslash$ ') such that for every $a, b, c \in G$, $ab = c \Leftrightarrow c/b = a \Leftrightarrow a \backslash c = b$. Functions are understood to operate on the right.

A homotopy of a quasigroup G_1 into a quasigroup G_2 with products φ_1, φ_2 is defined to be a triple $h = (\alpha, \beta, \nu)$ where $\alpha, \beta, \nu: G_1 \rightarrow G_2$ satisfy

$$(1) \quad (\alpha \times \beta)\varphi_2 = \varphi_1\nu$$

i.e.

$$\begin{array}{ccc} G_1 \times G_1 & \xrightarrow{\varphi_1} & G_1 \\ \alpha \times \beta \downarrow & & \downarrow \nu \\ G_2 \times G_2 & \xrightarrow{\varphi_2} & G_2 \end{array}$$

is a commuting diagram.

We note that (α, β, ν) is a homotopy of $(G_1, \cdot)$ into $(G_2, \cdot) \Leftrightarrow (\beta, \nu, \alpha)$ is a homotopy of $(G_1, /)$ into $(G_2, /) \Leftrightarrow (\alpha, \nu, \beta)$ is a homotopy of $(G_1, \backslash)$ into $(G_2, \backslash)$.

If α, β, ν are bijective, then h is said to be an isotopy. An isotopy of the form (α, β, I_G) of a quasigroup (G, φ_1) onto a quasigroup (G, φ_2) is said to be a principal isotopy.

A quasigroup G is said to be an abelian quasigroup if for every $a, b, c, d \in G$,

$$(2) \quad (ab)(cd) = (ac)(bd),$$

(Murdoch [6]).

For every a, b, c, d in an abelian quasigroup G

Received by the editors April 25, 1976 and, in revised form, July 27, 1976 and September 24, 1976.

AMS (MOS) subject classifications (1970). Primary 22A99.

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$$(3) \quad ab/cd = (a/c)(b/d),$$

$$(4) \quad ab \setminus cd = (a \setminus c)(b \setminus d),$$

$$(5) \quad (a/b) \setminus (c/d) = (a \setminus c)/(b \setminus d).$$

THEOREM 1. *For every element p in an abelian quasigroup $(G, \cdot)$, there exists a principal isotope (G, φ_p) of $(G, \cdot)$ which is a commutative group having p as the identity element; moreover for any two elements p, q in G , (G, φ_p) is isomorphic to (G, φ_q) .*

PROOF. $\varphi_p: G \times G \rightarrow G$ is defined by

$$(6) \quad (a, b)\varphi_p = (a(p \setminus (bp))) / p.$$

Then

$$\begin{aligned} (a, b)\varphi_p &= (a(p \setminus (bp))) / (p(p \setminus p)) \\ &= (a/p)((p \setminus (bp)) / (p \setminus p)) \quad [\text{by (3)}] \end{aligned}$$

$$= (a/p)((p/p) \setminus ((bp)/p)) \quad [\text{by (5)}]$$

$$(7) \quad = (a/p)((p/p) \setminus b)$$

$$(8) \quad = (a)R_p^{-1} \cdot (b)L_{p/p}^{-1}$$

where R_u (resp. L_u) denote the right (resp. left) translation by u in $(G, \cdot)$. Since $R_p^{-1}, L_{p/p}^{-1}$ are bijective, $(R_p^{-1}, L_{p/p}^{-1}, I_G)$ is a principal isotopy of $(G, \cdot)$ onto (G, φ_p) . Since an isotope of a quasigroup is a quasigroup (Bruck [2]), (G, φ_p) is a quasigroup.

p is the identity element in (G, φ_p) . Also (G, φ_p) is abelian. For if $a, b, c, d \in G$,

$$\begin{aligned} &((a, b)\varphi_p, (c, d)\varphi_p)\varphi_p \\ &= \left(((a/p)((p/p) \setminus b)) \left(p \setminus (((c(p \setminus (dp))))/p)p \right) \right) / p \\ &\quad [\text{by (6) and (7)}] \\ &= \left(((a/p)((p/p) \setminus b)) \left(((p/p)p) \setminus (c(p \setminus (dp))) \right) \right) / p \\ &= \left(((a/p)((p/p) \setminus b)) \left(((p/p) \setminus c)(p \setminus (p \setminus (dp))) \right) \right) / p \quad [\text{by (4)}] \\ &= \left(((a/p)((p/p) \setminus c)) \left(((p/p) \setminus b)(p \setminus (p \setminus (dp))) \right) \right) / p \quad [\text{by (2)}] \\ &= ((a, c)\varphi_p, (b, d)\varphi_p)\varphi_p. \end{aligned}$$

Since an abelian loop is a commutative group (Murdoch [6]), (G, φ_p) is a commutative group having p as the identity element and it is a principal isotope of $(G, \cdot)$.

Now if $p, q \in G$, then since both (G, φ_p) and (G, φ_q) are principally isotopic to $(G, \cdot)$ and the relation of principal isotopy is an equivalence

relation, (G, φ_p) is principally isotopic to (G, φ_q) . Since any two isotopic groups are isomorphic (G, φ_p) is isomorphic to (G, φ_q) .

REMARK 1. If p is an idempotent element of $(G, \cdot)$, then (8) gives

$$(9) \quad ab = (aR_p, bL_p)\varphi_p.$$

$R_p \in \text{Aut} \cdot (G, \varphi_p)$. For if $x, y \in G$, then

$$\begin{aligned} (x, y)\varphi_p R_p &= ((x, y)\varphi_p R_p, p)\varphi_p \\ &= (x, y)\varphi_p R_p R_p^{-1} \cdot (p)L_p^{-1} \\ &= (x/p \cdot p \setminus y) \cdot (p/p \cdot p \setminus p) \quad (\text{since } p \text{ is idempotent in } (G, \cdot)) \\ &= (xp/p)(p \setminus yp) \quad (\text{since } (G, \cdot) \text{ is abelian and } p \text{ is} \\ &\quad \text{idempotent in } (G, \cdot)) \\ &= (xR_p, yR_p)\varphi_p. \end{aligned}$$

Similarly $L_p \in \text{Aut} \cdot (G, \varphi_p)$. Also $R_p L_p = L_p R_p$. For, $(x)R_p L_p = p(xp) = (pp)(xp) = (px)(pp) = (px)p = (x)L_p R_p$.

This extends Murdoch's discussion on the last two pages of [6] by showing that the class of quasigroups $(G, \cdot)$ which can be obtained from any commutative group $(G, +)$ via an isotopy $(\alpha, \beta, I_G): (G, +) \rightarrow (G, \cdot)$ where $\alpha, \beta \in \text{Aut} \cdot (G, +)$ and $\alpha\beta = \beta\alpha$ is precisely the class of all abelian quasigroups having at least one idempotent element.

REMARK 2. If p is not an idempotent element of $(G, \cdot)$, then R_p, L_p are not necessarily automorphisms of (G, φ_p) . For example, consider the abelian quasigroup $(G, \cdot)$ given by the multiplication table

	1	2	3
1	2	1	3
2	1	3	2
3	3	2	1

If $p = 1$ (which is not an idempotent element of $(G, \cdot)$), then the abelian group (G, φ_p) is given by the multiplication table

	1	2	3
1	1	2	3
2	2	3	1
3	3	1	2

$R_p = L_p \notin \text{Aut} \cdot (G, \varphi_p)$. For, $(1, 2)\varphi_p R_p \neq (1R_p, 2R_p)\varphi_p$.

REMARK 3. A principal isotope of a commutative group need not be an abelian quasigroup. For example, let $G = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ be the cyclic group of order 4 and let α, β be the following permutations:

$$\alpha = \begin{pmatrix} \bar{0} & \bar{1} & \bar{2} & \bar{3} \\ \bar{1} & \bar{3} & \bar{2} & \bar{0} \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} \bar{0} & \bar{1} & \bar{2} & \bar{3} \\ \bar{0} & \bar{2} & \bar{1} & \bar{3} \end{pmatrix}.$$

If we define $a \times b = a\alpha + b\beta$, then $(G, \times)$ is a principal isotope of $(G, +)$, but since $(\bar{0} \times \bar{1}) \times (\bar{2} \times \bar{3}) \neq (\bar{0} \times \bar{2}) \times (\bar{1} \times \bar{3})$, it is not abelian.

LEMMA 1. Let $(G, +)$ be a commutative group and $f, g: G \rightarrow G$ two functions satisfying the following conditions:

(i) f and g are surjective.

(ii) The operation $(x, y) \rightarrow xy: G \times G \rightarrow G$ defined by $xy = (x)f + (y)g$ satisfies $(ab)(cd) = (ac)(bd)$.

Then by (i) there exist $p, q, r, s \in G$ such that $(p)f = 0, (q)g = 0, (r)f = q$ and $(s)g = p$; moreover, $(r)gf = -(s)fg$. If we set $C = (r)gf$ we have the following conclusions:

The functions $\varphi, \psi: G \rightarrow G$ defined by

$$(x)\varphi = (x + (r)g)f - C \quad \text{and} \quad (x)\psi = (x + (s)f)g + C$$

are commuting endomorphisms. Moreover they satisfy the relations

$$(x)f = (x)\varphi + \bar{p} \quad \text{and} \quad (x)g = (x)\psi + \bar{q}$$

where $\bar{p} = (0)f$ and $\bar{q} = (0)g$.

PROOF. From (ii) we have the functional equation

$$(10) \quad \begin{aligned} & ((a)f + (b)g)f + ((c)f + (d)g)g \\ &= ((a)f + (c)g)f + ((b)f + (d)g)g \quad \text{for all } a, b, c, d \in G. \end{aligned}$$

By (i) there exist $a, d \in G$ such that $(a)f = (d)g = 0$. Then taking $b = x, c = r$ from (10) we obtain

$$(11) \quad (x)gf = (r)gf + (x)fg \quad \text{for all } x \in G.$$

Similarly letting $c = x, b = s$ we obtain

$$(12) \quad (x)fg = (x)gf + (s)fg \quad \text{for all } x \in G.$$

From (11) and (12) we get that

$$(13) \quad (r)gf = -(s)fg = C \quad (\text{say}).$$

Let $x, y \in G$ be given. By (i) there exist $a, b, d \in G$ such that $(a)f = x, (b)g = y + (r)g, (d)g = 0$. Also let $c = r$. Then from (10) we obtain

$$(x + y + (r)g)f = (x + (r)g)f + (b)fg \quad \text{i.e.,}$$

$$(14) \quad (x + y + (r)g)f = (x + (r)g)f + (y + (r)g)f - C$$

[by (12) and (13)].

If we subtract C from both sides and recall the definition of φ we recognize that (14) shows that φ is an endomorphism.

We repeat this procedure this time with $(a)f = 0, (b)f = x + (s)f, (d)g = y, c = s$. Then (10) yields

$$\begin{aligned}
 (b)gf + ((s)f + y)g &= (x + y + (s)f)g \quad \text{i.e.,} \\
 (x + (s)f)g + C + ((s)f + y)g &= (x + y + (s)f)g \\
 &\quad [\text{by (11) and (13)}].
 \end{aligned}$$

Upon adding C on both sides and referring to the definition of ψ we conclude that ψ is an endomorphism.

In the defining equation for φ we replace x by $x - (r)g$ and obtain $(x)f = (x - (r)g)\varphi + C = (x)\varphi - (r)g\varphi + C$; taking $x = 0$ we find $\bar{p} = (0)f = -(r)g\varphi + C$. Analogously we deduce $(x)g = (x)\psi - (s)f\psi - C$ and $\bar{q} = (0)g = -(s)f\psi - C$. Thus for all $x \in G$

$$(15) \quad (x)f = (x)\varphi + \bar{p} \quad \text{and} \quad (x)g = (x)\psi + \bar{q}.$$

We use (15) to calculate

$$\begin{aligned}
 (x)gf &= ((x)\psi + \bar{q})f = (x)\psi\varphi + (\bar{q})\varphi + \bar{p} \quad \text{and} \\
 (x)fg &= ((x)\varphi + \bar{p})g = (x)\varphi\psi + (\bar{p})\psi + \bar{q}.
 \end{aligned}$$

Now by (11), $(r)gf = (x)gf - (x)fg = (x)[\varphi, \psi] + (\bar{q})\varphi + \bar{p} - (\bar{p})\psi - \bar{q}$ where $[\varphi, \psi] = \varphi\psi - \psi\varphi$. Thus $(x)[\varphi, \psi] = (0)[\varphi, \psi] = 0$ for all $x \in G$ i.e. φ and ψ commute.

LEMMA 2. *Let $(G, +)$ be a commutative group and $\bar{p}, \bar{q} \in G$ arbitrary. If $\varphi, \psi: G \rightarrow G$ are commuting surjective endomorphisms, and if we define $f, g: G \rightarrow G$ by $(x)f = (x)\varphi + \bar{p}$ and $(x)g = (x)\psi + \bar{q}$, then (i) and (ii) of Lemma 1 are satisfied.*

PROOF. One needs to check (ii):

$$\begin{aligned}
 (ab)(cd) &= ((a)\varphi + \bar{p} + (b)\psi + \bar{q})\varphi \\
 &\quad + \bar{p} + ((c)\varphi + \bar{p} + (d)\psi + \bar{q})\psi + \bar{q} \\
 &= (a)\varphi^2 + (b)\psi\varphi + (c)\varphi\psi + (d)\psi^2 + F(\bar{p}, \bar{q})
 \end{aligned}$$

and this expression is invariant against the permutation of b and c since φ and ψ commute.

We can now establish the following proposition.

PROPOSITION 1. *Let $(f, g, I_G): (G, +) \rightarrow (G, \cdot)$ be a principal isotopy from a commutative group to an abelian quasigroup. Then there is an isotopy $(\varphi, \psi, T_a): (G, \cdot) \rightarrow (G, +)$ where φ and ψ are commuting automorphisms of $(G, +)$ and $(x)T_a = x + a$.*

Specifically

$$\begin{aligned}
 (x)\varphi &= (x + (0)g^{-1}f^{-1}g)f - C = (x)f - (0)f, \\
 (x)\psi &= (x + (0)f^{-1}g^{-1}f)g + C = (x)g - (0)g, \\
 C &= (0)g^{-1}f^{-1}gf = -(0)f^{-1}g^{-1}fg, \quad a = -(0)f - (0)g.
 \end{aligned}$$

Conversely, if $(\varphi, \psi, T_a): (G, \cdot) \rightarrow (G, +)$ is an isotopy with a pair of commuting automorphisms φ, ψ of $(G, +)$ and $a \in G$, then for every pair $\bar{p}, \bar{q} \in G$ with $a + \bar{p} + \bar{q} = 0$ the functions $f, g: G \rightarrow G$ given by $(x)f = (x)\varphi + \bar{p}$ and $(x)g = (x)\psi + \bar{q}$ define a principal isotopy $(f, g, I_G): (G, +) \rightarrow (G, \cdot)$.

The proof follows from Lemmas 1 and 2.

Combining Theorem 1 and Proposition 1 we now deduce the following result:

THEOREM 2. *Every abelian quasigroup $(G, \cdot)$ is obtained from a commutative group $(G, +)$ which is a principal isotope of $(G, \cdot)$ via an isotopy $(\varphi, \psi, T_a): (G, \cdot) \rightarrow (G, +)$ where φ and ψ are commuting automorphisms of $(G, +)$ and $(x)T_a = x + a$.*

If $(G, \cdot)$ is a given abelian quasigroup and $p \in G$, then by Theorem 1, $(R_p^{-1}, L_{p/p}^{-1}, I_G): (G, \cdot) \rightarrow (G, +)$ is a principal isotopy onto a commutative group with neutral element p . Proposition 1 then allows an explicit calculation of φ, ψ and a in terms of p and the operations of $(G, \cdot)$ alone. Specifically

$$(x)\varphi = ((xp)p)/(p \setminus ((pp)p)), \quad (x)\psi = (p/p)x, \quad a = (pp)/(p \setminus ((pp)p)).$$

REMARK 4. If $(G, \cdot)$ is not an abelian quasigroup, then φ, ψ (defined in Proposition 1) are not necessarily automorphisms of $(G, +)$. For example, consider $(G, +), (G, \cdot)$ defined in Remark 1. In this case

$$\varphi = \begin{pmatrix} \bar{0} & \bar{1} & \bar{2} & \bar{3} \\ \bar{0} & \bar{2} & \bar{1} & \bar{3} \end{pmatrix} \quad \text{and} \quad \psi = \beta \cdot \varphi, \psi \notin \text{Aut} \cdot (G, +).$$

For, $(\bar{1} + \bar{3})\varphi = \bar{0} \neq \bar{1} = (\bar{1})\varphi + (\bar{3})\varphi$ and $(\bar{2} + \bar{3})\psi = \bar{2} \neq \bar{0} = (\bar{2})\psi + (\bar{3})\psi$.

A topological quasigroup (G, τ) is a quasigroup G endowed with a topology τ w.r. to which the operations of G are continuous.

An isotopy of a topological quasigroup (G_1, τ_1) onto a topological quasigroup (G_2, τ_2) with products φ_1, φ_2 is a triple $h = (\alpha, \beta, \nu)$ where $\alpha, \beta, \nu: (G_1, \tau_1) \rightarrow (G_2, \tau_2)$ are homeomorphisms and satisfy (1). An isotopy of the form (α, β, I_G) of a topological quasigroup (G, τ, φ_1) onto a topological quasigroup (G, τ, φ_2) is said to be a principal isotopy (Das [3]).

REMARK 5. All the above results are true if quasigroups, groups are replaced by topological quasigroups, topological groups and the terms isotopy, principal isotopy are used in the above sense.

REMARK 6. Since any topological property is an invariant of principal isotopy, topological properties possessed by a commutative topological group are also possessed by an abelian topological quasigroup.

THEOREM 3. *Let p be an arbitrary but fixed element of an abelian topological quasigroup (G, τ) . Then there exists a fundamental system $\mathcal{U}$ of closed neighbourhoods of p such that*

- (i) for every $U \in \mathcal{U}$, $U = (p(U \setminus (pp)))/p$;
- (ii) for every $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $(V(p \setminus (Vp)))/p \subset U$.

Conversely if G is an abelian quasigroup and if $\mathcal{Q}$ is a filter base in G satisfying (i), (ii), then there exists a unique topology τ in G such that (G, τ) is an abelian topological quasigroup and $(p)\tau = \mathcal{Q}$.

This follows from Theorem 1, Remark 5 and the standard characterization of a group topology through the filter of neighbourhoods of the identity.

I express my sincere thanks to the referee for the useful suggestions which improved the presentation of the paper.

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