THE DECIDABILITY OF THE THEORY OF BOOLEAN ALGEBRAS WITH THE QUANTIFIER "THERE EXIST INFINITELY MANY"

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ABSTRACT. By using the decidability of the weak second order theory of linear order we get the decidability of the theory of Boolean algebras with the additional quantifier Q_0 .

The quantifier Q_0 is defined as follows:

For any $\mathfrak{A}$,

$$\mathfrak{A} \models Q_0 x \varphi(x) \text{ iff } \text{card}(\{a \in \mathfrak{A} \mid \mathfrak{A} \models \varphi(a)\}) \geq \aleph_0.$$

Ba denotes the elementary theory of Boolean algebras, formulated in the language with the following nonlogical symbols:

one ternary predicate $Uxyz$ (expressing the fact, that z is the sum of x and y),

one binary predicate Cxy (expressing the fact, that y is the complement of x) and

two constants 0 and e (denoting respectively zero and unit).

For the sake of simplicity we assume that $0 \neq e$ is an axiom of Ba .

$Ba(Q_0)$ denotes the theory of Boolean algebras in the language of Ba , with the additional quantifier Q_0 .

LO denotes the elementary theory of linear order with least element, formulated in the language with the following nonlogical symbols:

one binary predicate $x < y$ (expressing the fact that x is less than y) and

one constant θ (denoting the least element).

LO^w denotes the weak second order theory of LO (that means, we add to the language the new variables $X, Y, Z, \dots$, ranging over finite sets, and the symbols $\in, \cup, \cap, \emptyset$, denoting respectively membership relation, union, intersection and empty set).

It is known that the weak second order theory of linear order is decidable (see [4] or [5]). Thus also LO^w is decidable.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras. $\mathfrak{A} \equiv_0 \mathfrak{B}$ denotes that $\mathfrak{A}$ and $\mathfrak{B}$ have the same theory in the language of Boolean algebras with the additional quantifier Q_0 .

EXAMPLE. Let $\mathfrak{B}_\omega$ be the Boolean algebra of finite and cofinite subsets of ω .

Received by the editors February 26, 1976.

AMS (MOS) subject classifications (1970). Primary 02G05; Secondary 02H10.

Key words and phrases. Decidability, Boolean algebra, quantifier "there exist infinitely many".

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Then it is easily seen (for instance by using Ehrenfeucht games) that $\mathfrak{B}_\omega \equiv (\mathfrak{B}_\omega)^2$. We set

$$\varphi(x) =_{\text{df}} Q_0 y Uxyx \quad \text{and} \quad \psi =_{\text{df}} \forall xy (Cxy \rightarrow \neg \varphi(x) \vee \neg \psi(y)).$$

$\varphi(x)$ expresses that there are infinitely many elements less than x and ψ expresses that for any element x , x or its complement cannot have infinitely many smaller elements. Then $\mathfrak{B}_\omega \models \psi$ and $(\mathfrak{B}_\omega)^2 \models \neg \psi$; thus we see that the theory $\text{Ba}(Q_0)$ is more expressive than Ba .

The following theorem can be found in [2] or [1, §13, Theorem 5.1]:

THEOREM 1. *Let $\mathfrak{A}$ be any Boolean algebra. Then there is a Boolean algebra $\mathfrak{B}$, with $\text{card}(\{a \mid a \in \mathfrak{B}\}) \leq \aleph_0$, such that $\mathfrak{A} \equiv_0 \mathfrak{B}$.*

Let $\mathfrak{M}$ be a linearly ordered set with first element. Then $\mathfrak{S}(\mathfrak{M})$ denotes the Boolean algebra, generated by the left-closed right-open intervals. One can show (see [3] or [6]):

THEOREM 2. *Let $\mathfrak{A}$ be a Boolean algebra with $\text{card}(\{a \mid a \in \mathfrak{A}\}) \leq \aleph_0$. Then there is a linearly ordered set $\mathfrak{M}$ such that $\mathfrak{A} \cong \mathfrak{S}(\mathfrak{M})$.*

Let $\mathfrak{M}$ be a linearly ordered set with first element and $x \in \mathfrak{S}(\mathfrak{M})$. Then there are elements $a_0, \dots, a_n, b_0, \dots, b_n \in \mathfrak{M}$ (or $a_0, \dots, a_n, b_0, \dots, b_{n-1} \in \mathfrak{M}$) with $a_0 < b_0 < \dots < a_n < b_n$ ($a_0 < b_0 < \dots < a_n$) such that

$$x = \{y \mid a_0 \leq y < b_0\} \cup \dots \cup \{y \mid a_n \leq y < b_n\} \\ (x = \{y \mid a_0 \leq y < b_0\} \cup \dots \cup \{y \mid a_n \leq y\}).$$

Thus x can be coded by the two disjoint finite sets $X_0 = \{a_0, \dots, a_n\}$ and $X_1 = \{b_0, \dots, b_n\}$ ($X_0 = \{a_0, \dots, a_n\}, X_1 = \{b_0, \dots, b_{n-1}\}$).

$X_0 \cup X_1$ is the *support* of x (denoted by $\text{supp } x$). We set

$$\begin{aligned} \text{Sup}(X, Y) =_{\text{df}} & X \cap Y = \emptyset \wedge \forall y (y \in Y \rightarrow \exists x (x \in X \wedge x < y)) \\ & \wedge \forall xy (x < y \wedge x \in X \wedge y \in X \\ & \rightarrow \exists z (z \in Y \wedge x < z \wedge z < y)) \\ & \wedge \forall xy (x < y \wedge x \in Y \wedge y \in Y \\ & \rightarrow \exists z (z \in X \wedge x < z \wedge z < y)); \end{aligned}$$

that means, X and Y are the code of some element of the corresponding Boolean algebra. It is possible to describe union, complement, zero and unit with the help of codes.

Let φ be any formula of the language of $\text{Ba}(Q_0)$. Then we have the following important fact:

$$\begin{aligned} \mathfrak{S}(\mathfrak{M}) \models Q_0 x \varphi(x) \\ \text{iff } \text{card}(\cup \{\text{supp } x \mid \mathfrak{S}(\mathfrak{M}) \models \varphi(x)\}) \geq \aleph_0. \end{aligned}$$

Now we are in the position to define a function $*$ from the set of formulas of $\text{Ba}(Q_0)$ to the set of formulas of LO^w such that for every sentence φ of $\text{Ba}(Q_0)$, $\text{Ba}(Q_0) \vdash \varphi$ iff $\text{LO}^w \vdash (\varphi)^*$.

$$(x = y)^* =_{\text{df}} X_0 = Y_0 \wedge X_1 = Y_1;$$

$$(x = o)^* =_{\text{df}} X_0 = \emptyset \wedge X_1 = \emptyset;$$

$$(x = e)^* =_{\text{df}} X_0 = \{\theta\} \wedge X_1 = \emptyset;$$

$$(Uxyz)^* =_{\text{df}} \forall x(x \in Z_0 \leftrightarrow \{[x \in X_0 \wedge \forall y(y \in Y_0 \wedge y < x \rightarrow \\ \exists z(z \in Y_1 \wedge y < z \wedge z < x))]\}$$

$$\vee [x \in Y_0 \wedge \forall y(y \in X_0 \wedge y < x \rightarrow$$

$$\exists z(z \in X_1 \wedge y < z \wedge z < x))]\});$$

$$\wedge \forall x(x \in Z_1 \leftrightarrow \{[x \in X_1 \wedge \forall y(y \in Y_0 \wedge y < x \rightarrow$$

$$\exists z(z \in Y_1 \wedge y < z \wedge z < x))]\}$$

$$\vee [x \in Y_1 \wedge \forall y(y \in X_0 \wedge y < x \rightarrow$$

$$\exists z(z \in X_1 \wedge y < z \wedge z < x))]\});$$

$$(Cxy)^* =_{\text{df}} \forall x(x \in X_0 \leftrightarrow x \in Y_1 \vee (x = \theta \wedge \theta \notin Y_0))$$

$$\wedge \forall x(x \in X_1 \leftrightarrow x \in Y_0 \wedge x \neq \theta);$$

$$(\neg \varphi)^* =_{\text{df}} \neg \varphi^*;$$

$$(\varphi \wedge \psi)^* =_{\text{df}} \varphi^* \wedge \psi^*;$$

$$(\exists x \varphi(x)) =_{\text{df}} \exists X_0 X_1 (\text{Sup}(X_0, X_1) \wedge (\varphi(x))^*);$$

$$(Q_0 x \varphi(x)) =_{\text{df}} \forall Y (\forall y(y \in Y \rightarrow \exists X_0 X_1 (\text{Sup}(X_0, X_1) \wedge \\ y \in X_0 \cup X_1 \wedge (\varphi(x))^*))$$

$$\rightarrow \exists Z (Y \neq Z \wedge \forall y(y \in Y \rightarrow y \in Z) \wedge$$

$$\forall y(y \in Z \rightarrow \exists X_0 X_1 (\text{Sup}(X_0, X_1) \wedge y \in X_0 \cup X_1 \wedge (\varphi(x))^*))).$$

Let φ be any sentence of the language of $\text{Ba}(Q_0)$. It follows immediately from the construction, that: if $\text{Ba}(Q_0) \vdash \varphi$, then $\text{LO}^w \vdash (\varphi)^*$. Together with Theorem 1 and Theorem 2 we also get: if $\text{LO}^w \vdash (\varphi)^*$, then $\text{Ba}(Q_0) \vdash \varphi$. Now it follows from the decidability of LO^w , that also $\text{Ba}(Q_0)$ is decidable.

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