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# EXTENSIONS OF TOTALLY PROJECTIVE GROUPS

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ABSTRACT. It is an unpublished observation of L. Fuchs and E. A. Walker that if $L$ is a fully invariant subgroup of the totally projective $p$-group $G$, then both $L$ and $G/L$ are totally projective. In this note we treat the more difficult converse question.

It is an unpublished observation of L. Fuchs and E. A. Walker that if $L$ is a fully invariant subgroup of the totally projective $p$-group $G$, then both $L$ and $G/L$ are totally projective. In this note we treat the more difficult converse question.

Recall that a reduced abelian $p$-group $G$ is said to be totally projective if $p^\alpha \operatorname{Ext}(G/p^\alpha G, K) = 0$ for all abelian groups $K$ and all ordinals $\alpha$. There are numerous characterizations of totally projective groups more enlightening than the definition [1, p. 90], but the main fact that we need is the following observation: If $G$ is totally projective, then so are $p^\alpha G$ and $G/p^\alpha G$ for all ordinals $\alpha$; and, conversely, if there is an ordinal $\alpha$ such that both $p^\alpha G$ and $G/p^\alpha G$ are totally projective, then $G$ is totally projective. It is, of course, this latter statement which we shall generalize.

We shall also require at a crucial juncture another characterization of totally projectives which may be viewed as a generalization of the classical Kulikov criterion. For this we need certain definitions. If $\lambda$ is a limit ordinal, we call a $p$-group $G$ a $C_\lambda$-group provided $G/p^\alpha G$ is totally projective for all $\alpha < \lambda$. Thus every abelian $p$-group is a $C_\omega$-group. Recall that the length of a reduced $p$-group $G$ is just the smallest ordinal $\lambda$ such that $p^\lambda G = 0$. We shall say that a reduced abelian $p$-group $G$ is $\sigma$-summable if its socle $G[p] = \{x \in G: px = 0\}$ is the ascending union of a sequence of subgroups $\{S_n\}_{n<\omega}$ where for each $n$ there is an ordinal $\alpha_n$ less than the length of $G$ such that $S_n \cap p^{\alpha_n} G = 0$. We can now state

THEOREM 1. *Let $\lambda$ be a limit ordinal cofinal with $\omega$. Then a $p$-group $G$ of length $\lambda$ is totally projective if and only if $G$ is a $\sigma$-summable $C_\lambda$-group.*

PROOF. Let $G$ be a totally projective group having limit length $\lambda$ cofinal with $\omega$. Then $G/p^\alpha G$ is totally projective for all $\alpha$ and thus $G$ is a $C_\lambda$-group. Since $G$ has limit length, it is a direct sum of groups of length strictly less than $\lambda$ [1, p. 97]. As $\lambda$ is cofinal with $\omega$, we actually have such a direct decomposition of $G$ into countably many summands $G = \bigoplus_{n<\omega} H_n$ where we may take

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$\alpha_n$ = length of $H_n$ strictly less than $\lambda$ and $\alpha_n < \alpha_{n+1}$. Taking $S_n$ to be the socle of $H_0 + \cdots + H_n$, we see that $G$ is $\sigma$-summable.

Conversely, suppose $G$ is a $\sigma$-summable $C_\lambda$-group of length $\lambda$. There are two distinct cases. First if $\lambda = \beta + \omega$ for some ordinal $\beta$, then the $\sigma$-summability of $G$ implies that $(p^\beta G)[p]$ is the ascending union of subgroups of bounded height (as computed in $p^\beta G$) and, hence, $p^\beta G$ is a direct sum of cyclic groups by the classical Kulikov criterion. But then we have both $p^\beta G$ and $G/p^\beta G$ totally projective and, therefore, $G$ is totally projective.

Finally, assume that $\lambda \neq \beta + \omega$ for all $\beta$ and let the $S_n$'s and $\alpha_n$'s be as in the definition of $\sigma$-summability. Since $\alpha_n + \omega < \lambda$ for all $n$, we may assume that each $\alpha_n$ is of the form $\beta_n + \omega$. Then by a routine argument we can enlarge each $S_n$ so that it is maximal disjoint in $G[p]$ from $p^{\alpha_n}G$ and still maintain the monotonicity of the sequence $\{S_n\}_{n<\omega}$. Indeed such an argument occurs in the standard proofs of the Kulikov criterion. Next we form an ascending sequence of subgroups $\{G_n\}_{n<\omega}$ where $G_n$ is maximal in $G$ with respect to having $S_n$ as its socle. Then $G = \bigcup_{n<\omega} G_n$ is totally projective by [3, Proposition 2.5].

Recall that the height $h_G(x)$ of a nonzero element $x$ in the reduced $p$-group $G$ is defined to be the least ordinal $\alpha$ such that $x \notin p^{\alpha+1}G$. We also set $h_G(0) = \infty$. Let $\boldsymbol{\alpha} = (\alpha_i)_{i<\omega}$ be an increasing sequence of ordinals and symbols $\infty$; that is, for each $i$, either $\alpha_i$ is an ordinal or $\alpha_i = \infty$ and $\alpha_i < \alpha_{i+1}$ provided $\alpha_i \neq \infty$. With each such sequence $\boldsymbol{\alpha}$ we associate the fully invariant subgroup $G(\boldsymbol{\alpha})$ of the $p$-group $G$ defined by

$$G(\boldsymbol{\alpha}) = \{x \in G\colon h_G(p^i x) \geqslant \alpha_i \text{ for all } i < \omega\}.$$

If $G$ is totally projective, then all of its fully invariant subgroups are of this form (see [1, Theorem 67.1 and Exercise 6, p. 101]). Thus we shall satisfy ourselves with showing that if for some sequence $\boldsymbol{\alpha}$ both $G(\boldsymbol{\alpha})$ and $G/G(\boldsymbol{\alpha})$ are totally projective, then $G$ itself is necessarily totally projective. We suspect that there does not exist a $p$-primary group $G$, necessarily not totally projective, that contains a fully invariant subgroup $L$ not of the form $G(\alpha)$ such that both $L$ and $G/L$ are totally projective.

Our first result will turn out to be a major step towards proving the desired theorem.

THEOREM 2. *Let $\lambda$ be a limit ordinal cofinal with $\omega$ and let $\boldsymbol{\alpha} = \{\alpha_i\}_{i<\omega}$ be an increasing sequence of ordinals bounded above by $\lambda$. If $G$ is a $C_\lambda$-group and if $G(\boldsymbol{\alpha})$ is totally projective, then $G$ is totally projective.*

PROOF. Let $L = G(\boldsymbol{\alpha})$. First we wish to show that it is enough to prove the theorem in the special case where (1) $\lambda = \sup \alpha_i$, (2) $\lambda$ is the length of $G$, and (3) $L$ is a direct sum of cyclic groups. Let $\delta = \sup \alpha_i \leqslant \lambda$. It is not difficult to verify that $p^\omega L = p^\delta G$ and, hence, $p^\delta G$ is totally projective. Therefore we need only establish that $G/p^\delta G$ is totally projective. But if $\delta < \lambda$, this follows from the assumption that $G$ is a $C_\lambda$-group. Hence we may assume that $\lambda = \delta = \sup \alpha_i$. Then it is easily seen that $L/p^\omega L = L/p^\lambda G = (G/p^\lambda G)(\boldsymbol{\alpha})$ and, of course, $L/p^{\omega}L$ is a direct sum of cyclic groups since $L$ is totally projective. Hence the desired reduction follows since $G/p^{\lambda}G$ is a $C_{\lambda}$-group if $G$ is.

Since now we are assuming $L$ to be a direct sum of cyclic groups, we have $L[p] = \bigcup_{n=1}^{\infty} T_n$ where $T_n \cap p^n L = 0$ and $T_n \subseteq T_{n+1}$ for each $n$. Now clearly $L[p] = (p^{\alpha_0}G)[p]$ and, hence, we have a direct decomposition $G[p] = S_0 + L[p]$ where $S_0 \cap p^{\alpha_0}G = 0$. It follows that $G[p]$ is the monotone union of the sequence of subgroups $\{S_n\}_{n<\omega}$ where $S_n = S_0 + T_n$ for $n \geqslant 1$. Moreover, $S_n \cap p^{\alpha_n}G = 0$ for each $n$ since

$$S_n \cap p^{\alpha_n}G = S_n \cap (p^{\alpha_n}G)[p] = S_n \cap (p^n L)[p] = S_n \cap p^n L = 0.$$

We conclude that $G$ is $\sigma$-summable and, therefore, Theorem 1 is applicable.

Next we turn to the very special case of our problem where $\alpha_0 = \alpha$ and $\alpha_i = \infty$ for $i > 1$.

Lemma. *If $L = (p^{\alpha}G)[p]$ and $G/L$ is totally projective, then $G$ is totally projective.*

Proof. First note that

$$p^{\alpha+1}G = p(p^{\alpha}G) \cong (p^{\alpha}G)/(p^{\alpha}G)[p] = p^{\alpha}(G/L)$$

and, hence, $p^{\alpha+1}G$ is totally projective. But this clearly implies that $p^{\alpha}G$ is totally projective, and it remains only to show that $G/p^{\alpha}G$ is also totally projective. Since $L \subseteq p^{\alpha}G$, we have $G/p^{\alpha}G \cong (G/L)/p^{\alpha}(G/L)$, and the latter group is totally projective since $G/L$ is.

Corollary. *If $L = G(\boldsymbol{\alpha})$ and $G/L$ is totally projective, then $G/pL$ is totally projective.*

Proof. Observe that $L/pL = p^{\alpha}(G/pL)[p]$ where $\alpha = \alpha_0$. Then $(G/pL)/(L/pL) \cong G/L$ is totally projective and we need only apply the lemma to $G/pL$.

We are now in position to prove

Theorem 3. *If $\boldsymbol{\alpha}$ is an increasing sequence of ordinals and symbols $\infty$ such that both $G(\boldsymbol{\alpha})$ and $G/G(\boldsymbol{\alpha})$ are totally projective $p$-groups, then $G$ itself is totally projective.*

Proof. Let $L = G(\boldsymbol{\alpha})$. If the sequence $\boldsymbol{\alpha}$ contains any symbols $\infty$, then there is a positive integer $n$ such that $p^n L = 0$. But then repeated applications of the above corollary yield the desired conclusion that $G$ is totally projective. Thus we may assume that $\boldsymbol{\alpha} = \{\alpha_i\}_{i<\omega}$ is an increasing sequence of ordinals and take $\lambda = \sup \alpha_i$. By Theorem 2, it suffices to show that $G$ is a $C_{\lambda}$-group. Hence we need only verify that $G/p^{\alpha_n}G$ is totally projective for all $n$. But since $p^n L \subseteq p^{\alpha_n}G$, we have

$$G/p^{\alpha_n}G \cong (G/p^n L)/p^{\alpha_n}(G/p^n L)$$

totally projective since each $G/p^n L$ is totally projective by our Corollary above.

## References

1. L. Fuchs, *Infinite Abelian groups*, Vol. II, Academic Press, New York, 1973.

2. C. K. Megibben, *The generalized Kulikov criterion*, Canad. J. Math. **21** (1969), 1192–1205. MR **40** #2754.

3. K. D. Wallace, *$C_\lambda$-groups and $\lambda$-basic subgroups*, Pacific J. Math. **43** (1972), 799–809. MR **47** #3569.

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