ROTA'S THEOREM FOR GENERAL FUNCTIONAL HILBERT SPACES

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ABSTRACT. By a theorem of G.-C. Rota, every (linear) operator T on a Hilbert space with spectral radius less than one is similar to the adjoint of the unilateral shift S of infinite multiplicity restricted to an invariant subspace. This theorem is shown to be true in a rather general context, where S is multiplication by z on a Hilbert space of functions analytic on an open subset D of the complex plane, and T is an operator with spectrum contained in D . A several-variable version for an N -tuple of commuting operators with a corollary concerning complete spectral sets is also presented.

1. Introduction. Some time ago, G.-C. Rota [11] showed that every (linear) Hilbert space operator T with spectral radius less than 1 is similar to the restriction of the adjoint of the unilateral shift S (of infinite multiplicity) to a suitable invariant subspace, and in this sense the unilateral shift is a "Universal Model" for such operators T . Later D. A. Herrero [10] and D. Voiculescu [14] obtained an analogous result with T an operator with spectrum contained in a "nice" bounded open subset D of the complex plane, and S multiplication by z on a certain Hilbert space of functions analytic on D . D.N. Clark [7] has obtained a several-variable version of the Rota theorem for N commuting strict contractions. In this paper, by a relatively simple kernel function argument similar to a technique appearing in work of C. A. Berger and B. I. Shaw [3] and of R. G. Douglas and C. Pearcy [8], we generalize the above results as follows:

THEOREM 1. *Let D be a bounded open subset of the complex plane. Let H be a Hilbert space of complex-valued functions on D (the vector space operations being the usual pointwise operations) such that*

- (1) *each element f in H is analytic on D ,*
- (2) *for each w in D , evaluation at w ($f \rightarrow f(w)$) is a bounded linear functional on H ,*
- (3) *the constant function 1 is in H , and*
- (4) *the operator $S: f(z) \rightarrow zf(z)$ is a bounded operator mapping H into itself.*

For K a Hilbert space, let $H_K = H \otimes K$ (the induced Hilbert space of K -valued functions), and let $S_K = S \otimes I_K$ on H_K . Then if T is an operator on K

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such that the spectrum $\sigma(T)$ of T is contained in D , then the adjoint of T is similar to the adjoint of S_K restricted to a suitable invariant subspace.

To state the next theorem, we recall a definition. If $T = (T_1, \dots, T_N)$ is an N -tuple of commuting operators on K , the joint spectrum $\sigma(T)$ of T is defined to be the set of all N -tuples $w = (w_1, \dots, w_N)$ in $\mathbb{C}^N$ such that $p(w)$ belongs to the spectrum of $p(T)$ for every multivariable polynomial $p = p(z_1, \dots, z_N)$ [2].

THEOREM 2. Let $D = D_1 \times \dots \times D_N$ be a bounded polyopen subset of $\mathbb{C}^N$ (i.e. each D_i is open in $\mathbb{C}$, $i = 1, \dots, N$) and suppose H is a Hilbert space of $\mathbb{C}$ -valued functions on D (the vector space operations the usual pointwise operations) such that

- (1) each element f in H is analytic on D ,
- (2) for each $w = (w_1, \dots, w_N)$ in D , evaluation at w ($f \rightarrow f(w)$) is a bounded linear functional on H ,
- (3) the constant function 1 is in H , and
- (4) for $j = 1, \dots, N$, the operator $S_j: f(z_1, \dots, z_N) \rightarrow z_j f(z_1, \dots, z_N)$ is a bounded operator mapping H into itself.

For K a Hilbert space, let $H_K = H \otimes K$ and $S_{K,j} = S_j \otimes I_K$ on H_K ($j = 1, \dots, N$). Then if $T = (T_1, \dots, T_N)$ is an N -tuple of commuting operators on K such that $\sigma(T) \subset D$, then T is similar to the adjoint of $S_K = (S_{K,1}, \dots, S_{K,N})$ restricted to a suitable invariant subspace; more precisely, there is a bounded invertible map Γ from K onto a subspace of H_K invariant under $S_{K,j}^*$ for each j , such that $S_{K,j}^* \Gamma = \Gamma T_j^*$.

2. Proof of Theorem 1. Let D , H and K be as in the hypotheses of Theorem 1. By (2) and the Riesz-Fréchet theorem there is a reproducing kernel function $k(\bar{w}, z)$, which, for each w in D , belongs to H as a function of z , and has the reproducing property

$$(a) \quad \langle f, k(\bar{w}, \cdot) \rangle_H = f(w).$$

The computation, for f in H and w in D ,

$$\langle f, S^* k(\bar{w}, \cdot) \rangle_H = \langle Sf, k(\bar{w}, \cdot) \rangle_H = wf(w) = \langle f, \bar{w} k(\bar{w}, \cdot) \rangle_H$$

shows

$$(b) \quad S^* k(w, \cdot) = \bar{w} k(w, \cdot) \text{ for } w \text{ in } D^* (= \{z: \bar{z} \in D\}).$$

Since $k(w, z) = \langle k(w, \cdot), k(\bar{z}, \cdot) \rangle_H = \overline{\langle k(\bar{z}, \cdot), k(w, \cdot) \rangle_H} = \overline{k(\bar{z}, \bar{w})}$, hypothesis (1) implies

$$(c) \text{ for each } z \text{ in } d, k(w, z) \text{ is analytic on } D^* \text{ as a function of } w.$$

Since $\sigma(T)$ is a compact subset of the open set D , there is a curve C contained in D , consisting of a finite number of rectifiable Jordan curves, and containing $\sigma(T)$ in its interior. Then C^* (the image of C under complex conjugation) is a curve contained in D^* and containing $\sigma(T^*)$ in its interior. By (c) we can define via the Riesz-Dunford functional calculus [7, VII.3], for each x in K , a K -valued function of z , $(\Gamma x)(z)$ by

$$\begin{aligned} (\Gamma x)(z) &= k(T^*, z)x = \left\{ \int_{C^*} k(w, z)(wI - T^*)^{-1} dw \right\} x \\ &= \int_{C^*} k(w, z)(wI - T^*)^{-1} x dw. \end{aligned}$$

Since $\|k(w, \cdot)\|_H^2 = k(w, \bar{w})$ is continuous on D^* , a compactness argument gives $\sup_{w \in C^*} \|k(w, \cdot)\|_H < \infty$. Similarly $\sup_{w \in C^*} \|(wI - T^*)^{-1}\| < \infty$, and hence,

$$\|\Gamma x\|_{H_K} \leq \sup_{w \in C^*} \|k(w, \cdot)\|_H \sup_{w \in C^*} \|(wI - T^*)^{-1}\| \|I(C^*)\| \|x\|.$$

Thus Γ is a bounded map of K into H_K .

Next we show Γ is bounded below. By hypothesis (3), the constant function $1 \in H$; let $\alpha = \|1\|_H > 0$. For x in K , $x \equiv 1 \otimes x$ is an element of H_K with $\|x\|_{H_K} = \alpha \|x\|_K$. Then

$$\begin{aligned} \langle x, \Gamma x \rangle_{H_K} &= \left\langle x, \int_{C^*} k(w, \cdot)(wI - T^*)^{-1} x dw \right\rangle_{H_K} \\ &= \int_C \langle (\bar{w}I - T)^{-1} x, k(w, \cdot)x \rangle_{H_K} d\bar{w} \\ &= \int_C \langle (\bar{w}I - T)^{-1} x, x \rangle_K d\bar{w} \quad \text{by (a)} \\ &= \langle x, x \rangle_K \quad \text{by the functional calculus for } T. \end{aligned}$$

Hence

$$\|x\|_K^2 = \langle x, \Gamma x \rangle_{H_K} \leq \|x\|_{H_K} \|\Gamma x\|_{H_K} = \alpha \|x\|_K \|\Gamma x\|_{H_K}$$

or $\|\Gamma x\|_{H_K} \geq \alpha^{-1} \|x\|_K$, and hence Γ is bounded below.

Finally, we compute

$$\begin{aligned} \Gamma T^* x &= \int_{C^*} k(w, \cdot)(wI - T^*)^{-1} (T^* x) dw \\ &= \int_{C^*} wk(w, \cdot)(wI - T^*)^{-1} x dw - \int_{C^*} k(w, \cdot)x dw \\ &= \int_{C^*} wk(w, \cdot)(wI - T^*)^{-1} x dw \quad \text{by (c)} \\ &= \int_{C^*} (S^* k(w, \cdot))(wI - T^*)^{-1} x dw \quad \text{by (b)} \\ &= S_K^* \Gamma x. \end{aligned}$$

Hence $\text{Ran } \Gamma$ is invariant under S_K^* , and T^* is similar to $S_K^*| \text{Ran } \Gamma$ via Γ .

3. Proof of Theorem 2. Let $D, H, K, T = (T_1, \dots, T_N)$ and $S = (S_1, \dots, S_N)$ be as in the statement of Theorem 2. Just as before, for each $w = (w_1, \dots, w_N)$ in D , there is a reproducing kernel function $k(\bar{w}, z)$ ($\bar{w} = (\bar{w}_1, \dots, \bar{w}_N)$) which belongs to H as a function of z and satisfies

(a) for f in H , w in D , $\langle f, k(\bar{w}, \cdot) \rangle_H = f(w)$,

(b) $S_j^* k(w, \cdot) = w_j k(w, \cdot)$ for w in D^* , and

(c) for each z in D , $k(w, z)$ is analytic on D^* as a function of w .

For $\lambda = (\lambda_1, \dots, \lambda_N)$ in $\mathcal{C}^N$, let $\pi_i: \lambda \rightarrow \lambda_i$ be the projection onto the i th coordinate. An elementary argument [2, p. 275] shows that $\pi_i(\sigma(T)) = \sigma(T_i)$. Since $D = D_1 \times \dots \times D_N$ is polyopen and $\sigma(T) \subset D$, it follows that $\sigma(T_i) \subset D_i$. Let $\mathcal{C}_i$ be a curve contained in D_i consisting of finitely many rectifiable Jordan curves and containing $\sigma(T_i)$ in its interior. Define $\Gamma: K \rightarrow H_K$ by

$$\Gamma x = \int_{C_N^*} \dots \int_{C_1^*} k(w, \cdot) (w_N I - T_N^*)^{-1} \dots (w_1 I - T_1^*)^{-1} x \, dw_1 \dots dw_N$$

The proof that Γ is bounded, bounded below, and that $\Gamma T_i^* = S_{K,i}^* \Gamma$ ($i = 1, \dots, N$) now proceeds exactly as in the proof of Theorem 1.

4. Examples. In this section we collect several examples of open sets D and corresponding spaces H satisfying conditions (1)–(4) of Theorem 1 or of Theorem 2.

EXAMPLE 1. Let D be a bounded domain of $\mathcal{C}$ and let H be the space of single-valued functions f analytic on D with modulus square-integrable with respect to area measure: $\iint_D |f(z)|^2 \, dx \, dy < \infty$. The verification of hypothesis (2) follows by an argument involving Taylor series [4, p. 5]; the others follow easily. For the case where D is the unit disk, the associated operator $S: f(z) \rightarrow zf(z)$ is the Bergman shift, and the kernel function $k(\bar{w}, z)$ is given by $\pi^{-1}(1 - \bar{w}z)^{-2}$.

EXAMPLE 2. Let D be a bounded domain in $\mathcal{C}$ whose boundary consists of a finite number of nonintersecting, analytic Jordan curves and let μ be any measure supported on ∂D which is boundedly mutually absolutely continuous with respect to arc-length measure on ∂D . Let H be the closure of $\text{Rat}(\bar{D})$ (rational functions with poles off of $\bar{D}$) in the space $L^2(\mu)$. An application of the Cauchy integral formula enables one to verify (1) and (2) for (D, H) [4, pp. 77–78], and (3) and (4) follow easily. This is essentially the situation considered by Herrero [10]. The corresponding operator S , for special choices of (D, H) , has been studied by Sarason [12] and Abrahamse and Douglas [1]. When D is the unit disk and μ is arc-length measure on ∂D , S is the unilateral shift and the kernel function $k(\bar{w}, z)$ is equal to $(1 - \bar{w}z)^{-1}$. In this case the map Γ in the proof of Theorem 1 specializes to that originally used by Rota [11].

The above construction works equally well with H chosen to be the closure of the polynomials in $L^2(\mu)$.

EXAMPLE 3. Let D be arbitrary bounded domain in $\mathcal{C}$. We show that there is a measure μ with support contained in $\partial \bar{D}$, such that, if H is the closure of $\text{Rat}(\bar{D})$ in $L^2(\mu)$, (D, H) satisfies (1)–(4). The argument again works equally with H chosen to be the closure of the polynomials in $L^2(\mu)$. The techniques are adapted from the work of Brennan [6].

Let t be some fixed point in D . By the maximum modulus theorem, evaluation at t is a bounded linear functional on $\text{Rat}(\bar{D})$ as a subspace of

$C(\partial\bar{D})$ (continuous functions on $\partial\bar{D}$ with sup norm), and hence, by the Hahn-Banach theorem and Riesz representation theorem, there is a measure μ supported on $\partial\bar{D}$ such that

$$f(t) = \int f(z) d\mu(z) \quad \text{for all } f \text{ in } \text{Rat}(\bar{D}).$$

Let H be the closure of $\text{Rat}(\bar{D})$ in $L^2(\mu)$ -norm, and let us say, for w a point in D , " w is a bounded evaluation" if evaluation at w is a bounded linear functional on $\text{Rat}(\bar{D})$ in $L^2(\mu)$ -norm.

Since $|f(t)| \leq \int |f(z)| d\mu(z) \leq \|f\|_{L^2(\mu)}$ for f in $\text{Rat}(\bar{D})$, t is a bounded evaluation. The operator S of multiplication by z on H is subnormal with minimal normal extension N equal to multiplication by z on $L^2(\mu)$, which has spectrum $\sigma(N) \subseteq \partial\bar{D}$. Since $(S - tI)H$ is in the kernel of the nontrivial bounded linear functional $f \rightarrow f(t)$ on H , $S - tI$ is not onto, and hence $t \in \sigma(S)$. By a result of Bram [5, Theorem 4], it follows that $D \subset \sigma(S)$.

Let $w \in D$ and suppose w is not a bounded evaluation. Let g be any function in $L^2(\mu)$ such that $\int fg d\mu = 0$ for all f in H . Then, in particular, for f in $\text{Rat}(\bar{D})$, $(f(z) - f(w))/(z - w)$ is also in $\text{Rat}(\bar{D})$, and hence

$$\int \frac{f(z) - f(w)}{z - w} g(z) d\mu(z) = 0,$$

or

$$(*) \quad \int f(z) \frac{g(z)}{z - w} d\mu(z) = f(w) \int \frac{g(z)}{z - w} d\mu(z).$$

Since $g(z)(z - w)^{-1} \in L^2(\mu)$ (w is away from $\text{supp } \mu$) and w is not a bounded evaluation, it follows that $\int [g(z)/(z - w)] d\mu(z) = 0$. Hence

$$(z - w)^{-1} \in H.$$

But this implies that multiplication by $(z - w)^{-1}$ is a bounded inverse for $(S - w)$ on H , contradicting that $w \in \sigma(S)$. Hence to each point w in D there corresponds a bounded linear functional λ_w on H such that $f(w) = \lambda_w(f)$ for all f in $\text{Rat}(\bar{D})$.

To show that $f(w) \equiv \lambda_w(f)$ is analytic on D for each f in H , it suffices to show (via a normal family argument) that $\|\lambda_w\|$ is uniformly bounded on compact subsets of D . This will follow if we show that, for any w_0 in D , $\|\lambda_w\|$ is uniformly bounded for w in some neighborhood of w_0 . For w_0 in D , an argument above gives $(z - w_0)^{-1} \notin H$. Hence there exists a $g \in L^2(\mu)$ such that $\int fg d\mu = 0$ for all f in H but $\int (z - w_0)^{-1} g(z) d\mu(z) = 1$. Then for f in $\text{Rat}(\bar{D})$, w in D , $(*)$ holds. But for w sufficiently close to w_0 ,

$$\left| \int (z - w)^{-1} g d\mu \right| > \frac{1}{2} \quad \text{and} \quad \|(z - w)^{-1} g\|_{L^2(\mu)} \leq 2 \|(z - w_0)^{-1} g\|_{L^2(\mu)}.$$

Hence, $(*)$ implies

$$|f(w)| = \left| \left\{ \int (z - w)^{-1} g \, d\mu \right\}^{-1} \int (z - w)^{-1} g f \, d\mu \right|$$

$$\leq 2 \|(z - w)^{-1} g\|_{L^2(\mu)} \|f\|_{L^2(\mu)} \leq 4 \|(z - w_0)^{-1} g\|_{L^2(\mu)} \|f\|_{L^2(\mu)}.$$

Hence (1) and (2) follow for (D, H) . Properties (3) and (4) are trivial.

EXAMPLE 4. For D a (not necessarily connected) bounded open set $\subseteq \mathbb{C}$, a space H corresponding to D can be constructed as follows. Let $\{D_i\}_{i=1}^N$ (N finite or ∞) be a listing of the components of D , and let H_i be a space associated with D_i as in any of the preceding examples. Let H be the space of functions analytic on D such that $f_i \equiv f|_{D_i}$ is in H_i for all i , with $\|f\|_H^2 = \sum \|f_i\|_{H_i}^2 / 2^i \|1\|_{H_i} < \infty$. Then it is easily verified that (D, H) satisfies (1)–(4).

In all of the above examples, the associated operator S has been subnormal. Examples of spaces (D, H) such that the associated S is not subnormal can be obtained by considering the $H^2(\beta)$ and $L^2(\beta)$ spaces associated with a nonsubnormal weighted shift operator, as discussed by Shields [13].

Examples where D is a polyopen subset of $\mathbb{C}^N$, $N > 1$, can be obtained by tensoring any combination of the above single-variable examples. The simplest example is obtained by taking D equal to the unit polydisk, μ the measure on the distinguished boundary equal to the N -fold Cartesian product of linear Lebesgue measure on the unit circle and H the closure of the polynomials in $L^2(\mu)$. In this case, the kernel function $k(\bar{w}, z)$ is equal to $(1 - \bar{w}_1 z_1)^{-1} \cdots (1 - \bar{w}_N z_N)^{-1}$, and the similarity mapping Γ in the proof of Theorem 2 is the same as that given by Clark [7].

5. Complete spectral sets. Recall, for $T = (T_1, \dots, T_N)$, an N -tuple of commuting operators on a Hilbert space K , a compact set $X \subseteq \mathbb{C}^N$ is called a *spectral set* for T if X contains $\sigma(T)$ and $\|f(T)\| \leq \sup\{|f(z)| : z \in X\}$ for every f in $\text{Rat}(X)$. The related but stronger notion of *complete spectral set* is defined in [2], where Arveson shows that a necessary and sufficient condition for X to be a complete spectral set for T is that $\sigma(T) \subseteq X$ and that T have a normal dilation $N = (N_1, \dots, N_N)$ (i.e., the N_i 's are commuting normal operators defined on a larger space K' such that $r(T) = P_K r(N)|_K$ for all $r \in \text{Rat}(X)$) with $\sigma(N) \subseteq \partial X$. Using this result and Theorem 2, we obtain

COROLLARY. Let $T = (T_1, \dots, T_N)$ be an N -tuple of commuting operators on a Hilbert space K , and let $D = D_1 \times \cdots \times D_N$ be a polyopen set containing $\sigma(T)$. Then $\bar{D}$ is a complete spectral set for an N -tuple $T' = (T'_1, \dots, T'_N)$ of commuting operators similar to T .

PROOF. For $i = 1, \dots, N$, let μ_i be a measure supported on $\partial \bar{D}_i$, such that, if H_i is the closure of $\text{Rat}(\bar{D}_i)$ in $L^2(\mu_i)$, (D_i, H_i) satisfies (1)–(4) in Theorem 1. (This can be arranged by combining Example 3 with Example 4.) Let μ be the measure $\mu_1 \times \cdots \times \mu_N$ supported on $\partial \bar{D}_1 \times \cdots \times \partial \bar{D}_N$ and let H be the closure of $\text{Rat}(\bar{D})$ in $L^2(\mu)$. Then (D, H) satisfies (1)–(4) in the statement of Theorem 2. Since $\sigma(T) \subseteq D$, Theorem 2 implies that T is similar to T' , where T' is the compression of S_K to some $*$ -invariant subspace. From this repre-

sentation, it is clear that T' has normal dilation $N = (N_1, \dots, N_N)$ consisting of multiplication by the coordinate functions on $L^2(\mu) \otimes K$. Since $\text{supp } \mu \subset \partial \bar{D}$, we have $\sigma(N) \subset \partial \bar{D}$, and hence $\bar{D}$ is a complete spectral set for T' .

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REFERENCES

1. M.B. Abrahamse and R.G. Douglas, *A class of subnormal operators related to multiply-connected domains*, Advances in Math. **19** (1976), 1–43.
2. W. B. Arveson, *Subalgebras of C^* -algebras. II*, Acta Math. **128** (1972), 271–308.
3. C.A. Berger and B.I. Shaw, *Intertwining, analytic structure, and the trace norm estimate*, Proc. Conf. on Operator Theory (Dalhousie Univ., Halifax, N.S., 1973), edited by P.A. Fillmore, Lecture Notes in Math., vol. 345, Springer-Verlag, Berlin and New York, 1973, pp. 1–12. MR **50** #14327.
4. S. Bergman, *The kernel function and conformal mapping*, Math. Surveys, no. 5, Amer. Math. Soc., Providence, R.I., 1950. MR **12**, 402.
5. J. Bram, *Subnormal operators*, Duke Math. J. **22** (1955), 75–94. MR **16**, 835.
6. J.E. Brennan, *Invariant subspaces and rational approximation*, J. Functional Analysis **7** (1971), 285–310.
7. D.N. Clark, *On commuting contractions*, J. Math. Anal. Appl. **32** (1970), 590–596. MR **42** #2309.
8. R.G. Douglas and C. Pearcy, *Invariant subspaces of non-quasitriangular operators*, Proc. Conf. on Operator Theory (Dalhousie Univ., Halifax, N.S., 1973), edited by P.A. Fillmore, Lecture Notes in Math., vol. 345, Springer-Verlag, Berlin and New York, 1973, pp. 13–57. MR **50** #10857.
9. N. Dunford and J.T. Schwartz, *Linear operators. I, General theory*, Interscience, New York, 1958. MR **22** #8302.
10. D.N. Herrero, *A Rota universal model for operators with multiply connected spectrum*, Rev. Roumaine Math. Pures Appl. **21** (1976), 15–23.
11. G.-C. Rota, *On models for linear operators*, Comm. Pure Appl. Math. **13** (1960), 469–472. MR **22** #2898.
12. D.E. Sarason, *The H^p spaces of an annulus*, Mem. Amer. Math. Soc. No. 55 (1965). MR **32** #6256.
13. A.L. Shields, *Weighted shift operators and analytic function theory*, Topics in Operator Theory, Math. Surveys, vol. 13, Amer. Math. Soc., Providence, R.I., 1974. MR **50** #14341.
14. D. Voiculescu, *Norm-limits of algebraic operators*, Rev. Roumaine Math. Pures Appl. **19** (1974), 371–378. MR **49** #7826.

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