ON A CHARACTERIZATION OF LOCALLY COMPACT GROUPS OF SECOND CATEGORY, ASSUMING THE CONTINUUM HYPOTHESIS

INDER K. RANA

ABSTRACT. Let G be a topological group of second category and having cardinality at most that of the continuum. Let $\mathbf{B}$ be some σ -algebra of subsets of G such that $(G, \mathbf{B})$ is a measurable group. For a probability measure P on $(G, \mathbf{B})$, write P_g for the measure defined by $P_g(E) = P(gE)$, $E \in \mathbf{B}$. The aim of this paper is to prove the following: if on $(G, \mathbf{B})$ there exists an inner-regular probability measure P such that $P_g \ll \mu$ for every $g \in G$, where μ is some σ -finite measure on $(G, \mathbf{B})$, then G is locally compact. Further if S denotes the σ -algebra generated by the topology of G and m denotes a Haar measure on G , then $\mu \gg m \gg P_g$ for every $g \in G$ on the σ -algebra $S \cap \mathbf{B}$.

1. Introduction. In [3] it was proved that given a standard measurable group $(G, \mathbf{B})$, P a probability measure and μ some σ -finite measure on $(G, \mathbf{B})$ such that $P_g \ll \mu$ for every $g \in G$, where $P_g(E) = P(gE)$ for every $E \in \mathbf{B}$, there exists a topology τ on G such that (G, τ) is a locally compact group, the σ -algebra generated by τ is $\mathbf{B}$ and if m denotes the Haar measure of (G, τ) , then $P_g \ll m \ll \mu$ for every $g \in G$. In the present paper we show, using the construction given in [3] and a result of [2], that topological groups of second category having cardinality at most that of the continuum (not necessarily standard) are locally compact if there exists an inner regular probability P on some σ -algebra $\mathbf{B}$ of subsets of G such that $(G, \mathbf{B})$ is a measurable group and $P_g \ll \mu$ for every $g \in G$, where μ is some σ -finite measure on $(G, \mathbf{B})$.

It may be remarked that we shall be assuming the continuum hypothesis.

2. Preliminaries. Let X be a topological space and μ a measure defined on some σ -algebra $\mathbf{B}$ of subsets of X . μ will be called *inner-regular* if $\mu(E) = \sup\{\mu(K) : K \in \mathbf{B}, K \subset E \text{ and } K \text{ compact}\}$, for every $E \in \mathbf{B}$. μ will be called *locally-finite* if for every $x \in X$ there exists some neighbourhood V_x of x such that $\mu(V_x) < \infty$.

G will stand for a topological group. G is said to be of second category if the topology of G is of second category. Let $\mathbf{B}$ be some σ -algebra of subsets of G . $(G, \mathbf{B})$ will be called a *measurable group* if the map $(g, h) \rightarrow gh^{-1}$ from $G \times G \rightarrow G$ is measurable when $G \times G$ is given the product σ -algebra. A measure μ on a measurable group $(G, \mathbf{B})$ is said to be (left) *quasi-invariant* if

Received by the editors July 6, 1976.

AMS (MOS) subject classifications (1970). Primary 43A05.

Key words and phrases. Measurable group, inner-regular measure, quasi-invariant measure.

© American Mathematical Society 1977

$\mu(E) = 0 \Leftrightarrow \mu(gE) = 0$ for every $g \in G$. Here $gE = \{gh: h \in E\}$. By the translates of a measure μ on a measurable group $(G, \mathbf{B})$, denoted by $\{\mu_g\}$, we mean the measures μ_g defined by $\mu_g(E) = \mu(gE)$ for every $E \in \mathbf{B}$ and $g \in G$.

With the above notations we shall prove the following:

THEOREM. *Let G be a topological group of second category and having cardinality at most that of the continuum. If $\mathbf{B}$ is some σ -algebra of subsets of G such that $(G, \mathbf{B})$ is a measurable group and if there exists an inner-regular probability measure P on $(G, \mathbf{B})$ such that $P_g \ll \mu$ for every $g \in G$, μ being some σ -finite measure on $(G, \mathbf{B})$, then G is locally compact. Moreover if S denotes the σ -algebra generated by the topology of G and m denotes the Haar measure of G , then $\mu \gg m \gg P_g$ for every $g \in G$ on the σ -algebra $S \cap \mathbf{B}$.*

Note that if G is any locally compact group and if we take $\mathbf{B} =$ the natural Borel σ -algebra on G and $P = \mu =$ some probability measure equivalent to the Haar measure of G , then all the conditions given in the theorem are satisfied.

For the proof of the theorem we need some elementary results. In the lemmas to follow X will denote a topological space and $\mathbf{B}$ some σ -algebra of subsets of X .

LEMMA 1. (i) *If μ is an inner-regular measure on $(X, \mathbf{B})$, then so is its restriction μ_B to any $B \in \mathbf{B}$, where $\mu_B(E) = \mu(B \cap E)$ for every $E \in \mathbf{B}$.*

(ii) *If $\{\mu_n\}$ is a sequence of inner-regular probability measures on $(X, \mathbf{B})$, then so is the measure ν defined by*

$$\nu(E) = \sum_{n=1}^{\infty} \frac{\mu_n(E)}{2^n} \quad \text{for every } E \in \mathbf{B}.$$

PROOF. (i) Let $E \in \mathbf{B}$ and let $\varepsilon > 0$ be given. Then by the inner regularity of μ , there exists a compact set $K \subset B \cap E$, $K \in \mathbf{B}$, such that $\mu(B \cap E) - \mu(K) < \varepsilon$. Since $K \subset E$ and $K \cap B = K$, we have

$$\begin{aligned} \mu_B(E) - \mu_B(K) &= \mu(B \cap E) - \mu(B \cap K) \\ &= \mu(B \cap E) - \mu(K) < \varepsilon, \end{aligned}$$

showing that μ_B is inner-regular.

(ii) Let $E \in \mathbf{B}$ and let $\varepsilon > 0$ be given. Since $\sum_{n=1}^{\infty} \mu_n(X)/2^n = 1$, we can choose a positive integer N such that $\sum_{n=N+1}^{\infty} \mu_n(X)/2^n < \varepsilon/2$. Now by inner-regularity of μ_n , $n = 1, 2, \dots, N$, there exist compact sets $K_n \in \mathbf{B}$, $K_n \subset E$, such that $\mu_n(E - K_n)/2^n < \varepsilon/2 \cdot N$, $n = 1, 2, \dots, N$. Put $K = \bigcup_{k=1}^N K_n$. Then K is compact, $K \in \mathbf{B}$ and $K \subset E$. Moreover,

$$\frac{\mu_n(E - K)}{2^n} \leq \frac{\mu_n(E - K_n)}{2^n} < \frac{\varepsilon}{2 \cdot N}, \quad n = 1, 2, \dots, N.$$

Then $\sum_{n=1}^N \mu_n(E - K)/2^n < \varepsilon/2$. Now

$$\begin{aligned} \nu(E - K) - \sum_{n=1}^N \frac{\mu_n(E - K)}{2^n} &= \sum_{n=1}^{\infty} \frac{\mu_n(E - K)}{2^n} - \sum_{n=1}^N \frac{\mu_n(E - K)}{2^n} \\ &= \sum_{n=N+1}^{\infty} \frac{\mu_n(E - K)}{2^n} \leq \sum_{n=N+1}^{\infty} \frac{\mu_n(K)}{2^n} < \frac{\varepsilon}{2}. \end{aligned}$$

Thus $\nu(E - K) < \sum_{n=1}^N \mu_n(E - K)/2^n + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Thus given $E \in \mathbf{B}$ and $\varepsilon > 0$, we have found a compact set $K \in \mathbf{B}$, $K \subset E$, such that $\nu(E - K) < \varepsilon$, which proves that ν is inner-regular.

LEMMA 2. *If μ and ν are two probability measures on $(X, \mathbf{B})$, ν inner-regular, then there exist measures ν_1 and ν_2 , both inner-regular, such that $\nu = \nu_1 + \nu_2$, $\nu_1 \perp \mu$ and $\nu_2 \ll \mu$.*

PROOF. Follows immediately from the proof of the well-known Lebesgue decomposition theorem (for example, the one given in [1]) keeping in view Lemma 1. We will call $\nu = \nu_1 + \nu_2$ the Lebesgue decomposition of the inner-regular measure ν with respect to μ .

For the proof of the theorem, we will also be requiring the following result due to Kai-Wang Ng, for a proof of which we refer to [2]. The result is as follows:

“Let G be a topological group of second category. Then there exists a locally finite inner-regular, quasi-invariant measure on a σ -field $\mathbf{B}$ of subsets of G , such that $(G, \mathbf{B})$ is a measurable group, if and only if G is locally compact.”

3. Proof of the Theorem. We first prove that under the given hypothesis there exists a quasi-invariant inner-regular probability measure on $(G, \mathbf{B})$. To start with, note that since P is inner-regular, so is P_g for every $g \in G$. Further we can assume without any loss of generality that μ is a probability measure. In case G is a countable group say $G = \{g_i\}$, it is obvious that

$$\nu(E) = \sum_{i=1}^{\infty} \frac{P_{g_i}(E)}{2^i}, \quad E \in \mathbf{B},$$

will be the required quasi-invariant, inner-regular probability measure. So let the cardinality of G be that of the continuum. We choose a well-ordering ‘<’ of elements of G such that for every $g \in G$, the set $\{h: h < g\}$ is countable (that such a well-ordering exists, having assumed the continuum hypothesis, we refer to p. 29 of General Topology by Kelley). Let the elements of G be written as a transfinite sequence $g_1, g_2, \dots, g, \dots$. Put $P_{g_1}^1 = P_{g_1}$, $P_{g_1}^2 = 0$. Let $P_{g_2} = P_{g_2}^1 + P_{g_2}^2$ be the Lebesgue decomposition of P_{g_2} with respect to $P_{g_1}^1$ (given by Lemma 2) where $P_{g_2}^1 \perp P_{g_1}^1$, $P_{g_2}^2 \ll P_{g_1}^1$ and both $P_{g_2}^1, P_{g_2}^2$ are inner-regular. Let $P_{g_3} = P_{g_3}^1 + P_{g_3}^2$ be the Lebesgue decomposition of P_{g_3} with respect to $P_{g_1}^1 + P_{g_2}^1$ (given by Lemma 2). In general suppose P_h^1 and P_h^2 have been defined for every $h < g$. Let $P_g = P_g^1 + P_g^2$ be the Lebesgue decomposition of P_g (given by Lemma 2) with respect to $\sum_{h < g} C_h^g P_h^1$ where for every g ,

$\{C_h^g\}_{h < g}$ is some sequence of positive numbers such that $\sum_{h < g} C_h^g P_h^1$ is a probability measure. Then by transfinite induction we get the families $\{P_g^1\}$ and $\{P_g^2\}$ with the following properties:

- (i) $P_g = P_g^1 + P_g^2$.
- (ii) $P_g^1 \perp \sum_{h < g} C_h^g P_h^1$, $P_g^2 \ll \sum_{h < g} C_h^g P_h^1$.
- (iii) P_g^1 and P_g^2 are inner-regular.

Consider the family $\{P_g^1\}$. Note that for $g \neq h$, $P_g^1 \perp P_h^1$ and $P_g^1 \ll \mu$ for every $g \in G$. Thus $\{P_g^1\}$ is a family of measures of cardinality that of the continuum, its members are mutually orthogonal and are dominated by μ . Thus $\{P_g^1\}$ satisfies the conditions of Lemma 2 of [3] and hence is countable say $\{P_{g(i)}^1\}$. Define $\nu(E) = \sum_{i=1}^{\infty} C_i P_{g(i)}^1(E)$, $E \in \mathbf{B}$ (where C_i is some sequence of positive numbers such that ν is a probability measure). Then it is easy to see that ν is quasi-invariant and it follows from Lemma 1 that ν is inner-regular also.

Thus under the given hypothesis on $(G, \mathbf{B})$ we have constructed an inner-regular quasi-invariant probability measure ν on $(G, \mathbf{B})$. Now we apply the result of Kai-Wang Ng, and get that G is locally compact.

Let S denote the σ -algebra generated by the open subsets of G . Then $S \cap \mathbf{B}$ is a σ -algebra and is invariant under the group operations. Let m denote the Haar measure of G . Then m and ν are two quasi-invariant measures on $(G, S \cap \mathbf{B})$ and thus they must be equivalent (see [4, Theorem 3.1.5]). Thus $m \equiv \nu \gg P_g$ on $S \cap \mathbf{B}$ for every $g \in G$. Obviously from the construction of ν it follows that $\mu \gg \nu$ and thus $\mu \gg m \gg P_g$ on $S \cap \mathbf{B}$ for every $g \in G$. This completes the proof of the theorem.

REFERENCES

1. P. R. Halmos, *Measure theory*, Van Nostrand, Princeton, N. J., 1950. MR 11, 504.
2. Kai Wang Ng, *Quasi-invariant measures on groups of second category*, J. London Math. Soc. 7 (1973), 170–174.
3. I. K. Rana, *On a characterization of standard measurable groups*, Sankhya (to appear).
4. Xia-Dao Xing, *Measure and integration theory on infinite-dimensional spaces. Abstract Harmonic analysis*, Academic Press, New York, 1972. MR 46 #9281.

INDIAN STATISTICAL INSTITUTE, 7, S.J.S. SANSANWAL MARG, NEW DELHI 110029, INDIA