FOUR METRIC CONDITIONS CHARACTERIZING ČECH DIMENSION ZERO

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ABSTRACT. If (X, d) is a metric space let $d_x(y) = d(x, y)$. It is proved that if each x in X has a neighbourhood P with $d_x(P)$ not dense in any neighbourhood of 0 in $[0, \infty)$ then $\text{Ind } X = 0$. This metric condition characterizes metrizable spaces which have Čech dimension zero. Three other metric characterizations are given.

Introduction. In [1], [2] and [3] metric conditions sufficient for a metrizable topological space to have Čech dimension zero are given. In [3, Theorem 3.4, p. 66] it is shown that if a space X has a compatible metric d with range not dense in any neighbourhood of 0 in $[0, \infty)$ then $\text{Ind } X = 0$. In this paper it is shown that the conclusion is the same when the condition on the metric holds only "locally" or "pointwise". Specifically, if d satisfies for each x in X there is a neighbourhood P of x such that $d_x(P)$ is not dense in any neighbourhood of 0 in $[0, \infty)$ then $\text{Ind } X = 0$. Conversely, if a metrizable space has Čech dimension zero then there exists a compatible metric satisfying the given condition globally and hence locally. Three other related characterizations are given.

First two preliminary constructions are made. Let (X, d) be a metric space. If $A \subset X$ is not empty let $d_A(x) = \inf \{d(a, x) : a \in A\}$ for each x in X . Let $d_x = d_{\{x\}}$. For each n in $\mathbb{N}$ let

$$E(n) = \{x \in X : d_x(X) \text{ is not dense in } [2^{-n-1}, 2^{-n}]\}.$$

Then each $E(n)$ is open.

For each pair of rational numbers r and s with $0 < r < s$ let

$$A(r, s) = \{x \in X : d_x(X) \cap [r, s] = \emptyset\}.$$

For each rational ϵ with $r < \epsilon < s$ let

$$\mathcal{B}(r, \epsilon, s) = \{B(x, \epsilon) : x \in A(r, s)\}$$

where $B(x, \epsilon)$ is the metric ball of center x and radius ϵ . The family $\mathcal{B}(r, \epsilon, s)$ is closure preserving and consists of sets which are closed and open. Well order $A(r, s)$ and, for each x in $A(r, s)$ and ϵ in $\mathbb{Q}$ with $r < \epsilon < s$, let

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$$D(x, \varepsilon) = B(x, \varepsilon) \setminus \bigcup_{y < x} B(y, \varepsilon).$$

Let

$$\mathcal{D}(r, \varepsilon, s) = \{D(x, \varepsilon) : x \in A(r, s)\}.$$

The family $\mathcal{D}(r, \varepsilon, s)$ is closure preserving and consists of disjoint closed and open sets. Finally let

$$\mathfrak{B} = \{\mathcal{D}(r, \varepsilon, s) : 0 < r < \varepsilon < s \text{ and } r, \varepsilon, s \in \mathbb{Q}\}.$$

Then $\mathfrak{B}$ is a σ -locally finite family of subsets of X which are closed and open.

THEOREM 1. *Let (X, d) be such that each x in X has a neighbourhood P satisfying $d_x(P)$ is not dense in any neighbourhood of 0 in $[0, \infty)$. Then $\text{Ind } X = 0$.*

PROOF. Under the given conditions the family $\mathfrak{B}$ is a σ -locally finite base for X consisting of sets which are closed and open. Thus, by [4, Theorem 5, p. 291], $\text{Ind } X = 0$.

The hypothesis of Theorem 1 may be restated as follows: each x in X is in $E(n)$ for an infinite number of integers n in $\mathbb{N}$.

THEOREM 2. *Let $E(n)$ be dense in X for an infinite number of positive integers n . Then $\text{Ind } X = 0$.*

PROOF. The proof consists in showing that $\mathfrak{B}$ is a base for X . Let P be open and let $x \in P$. Let n in $\mathbb{N}$ satisfy $B(x, 2^{-n+1}) \subset P$ and $E(n)$ is dense in X . Let $y \in E(n) \cap B(x, 2^{-n-1})$. Then there exist rational numbers r, s, ε , with $2^{-n-1} < r < \varepsilon < s < 2^{-n}$ and such that $y \in A(r, s)$. Because $x \in B(y, \varepsilon)$ it follows that there is a z in $A(r, s)$ with $x \in D(z, \varepsilon) \subset B(y, \varepsilon) \subset B(x, 2^{-n+1}) \subset P$. This completes the proof.

THEOREM 3. *If $d_F(X)$ is not a neighbourhood of 0 in $[0, \infty)$ for all closed subsets F in X then $\text{Ind } X = 0$.*

PROOF. Let $P \subset X$ be any open subset. Then there exists a sequence $a_n \downarrow 0$ in $[0, \infty)$ such that $a_n \notin d_{X-P}(X)$ for each n in $\mathbb{N}$.

Let $G_i = \{x \in X : d_{X-P}(x) > a_i\}$. Then $P = G_1 \cup (G_2 \setminus G_1) \cup (G_3 \setminus G_2) \cup \dots$. The sets in this union are closed, open and disjoint. Since every open set has such a decomposition it follows from [4, Theorem 5, p. 291] that $\text{Ind } X = 0$.

THEOREM 4. *Let (X, d) be a metric space and suppose that there exists a sequence (d_n) of continuous metrics on X converging uniformly to d and such that each d_n satisfies either the hypothesis of Theorem 1 or of Theorem 2. Then $\text{Ind } X = 0$.*

PROOF. Let $X_n = (X, d_n)$ and consider the mapping

$$f: X \rightarrow \prod_{n=1}^{\infty} X_n = Y$$

defined by $x \rightarrow (x, x, x, \dots)$. The map f is an embedding. By Theorems 1 and 2, for each n , $\text{Ind } X_n = 0$. Hence $\text{Ind } Y = 0$. Therefore $\text{Ind } X = 0$ since Y is metrizable. This completes the proof.

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