A REMARKABLE CLASS OF CONTINUED FRACTIONS¹

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ABSTRACT. For any irrational number α and integer $a > 1$, the continued fraction of $(a-1)\sum_{r=1}^{\infty} 1/a^{[ra]}$ is computed explicitly in terms of the continued fraction of α .

1. Introduction. In [2], the second author proved the remarkable result that $\sum_{r=1}^{\infty} 1/2^{[ra]}$ where $\alpha = (1 + \sqrt{5})/2$ has as its continued fraction $[0, 1, t_2, t_3, \dots]$ where $t_n = 2^{f_n-2}$ ($n \geq 2$) and f_n is the n th Fibonacci number. In this paper we generalize this result.

Let $\alpha > 0$ be any real number and let $a > 1$ be an integer.² Define

$$(1) \quad S_a(\alpha) = (a-1) \sum_{r=1}^{\infty} \frac{1}{a^{[ra]}}.$$

Consider the continued fraction for α^{-1} ,

$$\alpha^{-1} = [a_0; a_1, a_2, \dots].$$

(We use the usual continued fraction notation, say for example in [3].) Let p_n/q_n denote the convergents of α^{-1} . We adopt the convention throughout this paper that $p_{-2} = 0, p_{-1} = 1, q_{-2} = 1, q_{-1} = 0$. Then for all $n \geq 0$,

$$p_n = a_n p_{n-1} + p_{n-2} \quad \text{and} \quad q_n = a_n q_{n-1} + q_{n-2}.$$

Define another sequence of integers

$$(2) \quad t_0 = a_0 a, \quad t_n = (a^{q_n} - a^{q_{n-2}}) / (a^{q_{n-1}} - 1) \quad \text{for } n \geq 1.$$

(t_n is an integer since $q_{n-1} | q_n - q_{n-2}$.)

THEOREM. If α is irrational and $\alpha > 0$, then

$$S_a(\alpha) = [t_0; t_1, t_2, \dots].$$

We will also give explicit formulas for convergents P_n/Q_n of $S_a(\alpha)$. For this we need to define for real x ,

$$[x]' = \begin{cases} [x], & \text{if } x \text{ is not an integer,} \\ x-1, & \text{if } x \text{ is an integer,} \end{cases}$$

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² Any a with $|a| > 1$ yields the same result.

the largest integer $< x$. Further define

$$(3) \quad S'_a(\alpha) = (a-1) \sum_{r=1}^{\infty} \frac{1}{a^{[r\alpha]}}.$$

(Of course, for irrational α , $S_a(\alpha) = S'_a(\alpha)$.)

PROPOSITION. *If α is irrational, p_n/q_n are the convergents of α^{-1} and P_n/Q_n are the convergents of $S_a(\alpha)$, then for $n \geq 0$,*

$$\frac{P_n}{Q_n} = \begin{cases} S'_a(q_n/p_n), & n \text{ even}, \\ S_a(q_n/p_n), & n \text{ odd} \end{cases}$$

(if $p_0 = 0$ interpret $S'_a(q_0/p_0) = 0$).

Also

$$(4) \quad P_n = \begin{cases} \sum_{i=1}^{p_n} a^{q_n - [iq_n/p_n]}, & n \text{ even}, \\ \sum_{i=1}^{p_n} a^{q_n - [iq_n/p_n]}, & n \text{ odd}, \end{cases}$$

and

$$(5) \quad Q_n = (a^{q_n} - 1)/(a - 1).$$

In the last section we observe that $S_a(\alpha)$ is transcendental.

2. Lemmas. We assume throughout that α is irrational.

LEMMA 1. *Assume $1 \leq r < p_{n+1}$ and $n \geq 0$. Then*

$$[r\alpha] = \begin{cases} [rq_n/p_n]', & n \text{ even}, \\ [rq_n/p_n], & n \text{ odd} \end{cases}$$

(for $n = 0$ assume $p_0 \neq 0$).

PROOF. The n th convergent to α^{-1} is p_n/q_n and thus q_n/p_n is the $n+1$ or $n-1$ (depending on whether $a_0 = 0$ or $a_0 \neq 0$, respectively) convergent of α . Thus

$$\alpha = q_n/p_n + \theta_n/p_n p_{n+1},$$

where $0 < \theta_n < 1$ if n is odd and $-1 < \theta_n < 0$ if n is even. Hence, $1 \leq r < p_{n+1}$ implies

$$r\alpha = rq_n/p_n + \theta'_n/p_n,$$

where $0 < \theta'_n < 1$ if n is odd and $-1 < \theta'_n < 0$ if n is even. If $p_n \nmid r$, then there is no integer between rq_n/p_n and $r\alpha$ and

$$[r\alpha] = [rq_n/p_n] = [rq_n/p_n]'.$$

If $p_n \mid r$, then

$$[r\alpha] = rq_n/p_n + [\theta'_n/p_n],$$

which immediately gives the desired result.

LEMMA 2. Assume $1 \leq r < p_n$ and $n \geq 0$. Then

$$\begin{aligned} [rq_n/p_n] &= [rq_{n-1}/p_{n-1}]', & n \text{ odd}, \\ [rq_n/p_n]' &= [rq_{n-1}/p_{n-1}], & n \text{ even}. \end{aligned}$$

PROOF. Immediate from Lemma 1.

LEMMA 3.

$$\lim_{\substack{n \rightarrow \infty \\ n \text{ odd}}} S_a \left(\frac{q_n}{p_n} \right) = \lim_{\substack{n \rightarrow \infty \\ n \text{ even}}} S_a' \left(\frac{q_n}{p_n} \right) = S_a(\alpha).$$

PROOF. If n is even, we have by Lemma 1,

$$\left| S_a' \left(\frac{q_n}{p_n} \right) - S_a(\alpha) \right| = (a-1) \left| \sum_{r=p_{n+1}}^{\infty} (a^{-[rq_n/p_n]'} - a^{-[ra]}) \right|$$

and it is trivial to see that this tends to zero. The situation is similar if n is odd.

LEMMA 4.

$$(i) \quad S_a \left(\frac{q_n}{p_n} \right) = \frac{a-1}{a^{q_n}-1} \sum_{t=1}^{p_n} a^{q_n - [tq_n/p_n]}.$$

$$(ii) \quad S_a' \left(\frac{q_n}{p_n} \right) = \frac{a-1}{a^{q_n}-1} \sum_{t=1}^{p_n} a^{q_n - [tq_n/p_n]'}$$

PROOF. In definition (3) of $S_a'(q_n/p_n)$, set $r = kp_n + t$, $1 \leq t \leq p_n$, $0 \leq k < \infty$. Thus

$$\begin{aligned} S_a' \left(\frac{q_n}{p_n} \right) &= (a-1) \sum_{k=0}^{\infty} \sum_{t=1}^{p_n} a^{-[kq_n + tq_n/p_n]'} \\ &= (a-1) \sum_{t=1}^{p_n} a^{-[tq_n/p_n]'} \sum_{k=0}^{\infty} a^{-kq_n} \\ &= (a-1) \left(\sum_{t=1}^{p_n} a^{-[tq_n/p_n]'} \right) \left(1 - \frac{1}{a^{q_n}} \right)^{-1} \end{aligned}$$

which is the desired result. $S_a(q_n/p_n)$ is similar.

3. Proof of the theorem³ and proposition. From Lemmas 3 and 4 we see that it suffices to show that the quantities on the right-hand sides of (4) and (5) satisfy the appropriate recursion relations. Let P'_n denote the right side of (4) and Q'_n denote the right side of (5) ($n \geq 0$).

Set $Q'_{-2} = 1$, $Q'_{-1} = 0$. We need that for all $n \geq 0$,

$$Q'_n = t_n Q'_{n-1} + Q'_{n-2},$$

where t_n is given in (2). For $n = 0$ this is clear. For $n = 1$,

³ This theorem was already proved by Loxton and Van der Poorten (see [4], [5]). Their proof resembles, in much less explicit form, our proof for the continued fraction identity.

$$t_1 Q'_0 + Q'_{-1} = t_1 = (a^{q_1} - a^{q-1}) / (a^{q_0} - 1) = Q'_1.$$

For $n \geq 2$, we have by induction

$$\begin{aligned} t_n Q'_{n-1} + Q'_{n-2} &= \frac{a^{q_n} - a^{q_{n-2}}}{a^{q_{n-1}} - 1} \frac{a^{q_{n-1}} - 1}{a - 1} + \frac{a^{q_{n-2}} - 1}{a - 1} \\ &= (a^{q_n} - 1) / (a - 1) = Q'_n \end{aligned}$$

as desired.

Set $P'_{-2} = 0$, $P'_{-1} = 1$. We need that for all $n \geq 0$,

$$(6) \quad P'_n = t_n P'_{n-1} + P'_{n-2}.$$

If $n = 0$, then $t_n P'_{n-1} + P'_{n-2} = t_0 = a_0 a$, whereas

$$P'_0 = \sum_{t=1}^{p_0} a^{q_0 - [tq_0/p_0]'} = \sum_{t=1}^{a_0} a^{1 - [t/a_0]'} = a_0 a.$$

(If $a_0 = 0$, this is still valid.)

If $n = 1$,

$$t_1 P'_0 + P'_{-1} = a_0 a \frac{a^{q_1} - a^{q-1}}{a^{q_0} - 1} + 1 = a_0 a \frac{a^{a_1} - 1}{a - 1} + 1.$$

Also

$$\begin{aligned} P'_1 &= \sum_{t=1}^{p_1} a^{q_1 - [tq_1/p_1]'} = \sum_{t=1}^{p_1-1} a^{q_1 - [tq_0/p_0]'} + 1 \\ &= \sum_{t=1}^{p_1-1} a^{a_1 - [t/a_0]'} + 1, \end{aligned}$$

using Lemma 2. (If $a_0 = 0$ the result is clear.) It is easily seen that the numbers $[t/a_0]'$, $1 \leq t \leq p_1 - 1 = a_1 a_0$, give a_0 repetitions of each of the numbers $0, 1, 2, \dots, a_1 - 1$. Hence

$$P'_1 = a_0 \sum_{j=0}^{a_1-1} a^{a_1-j} + 1$$

which is the same as (7).

Now suppose $n > 1$. We verify (6) in the case that n is even; the case where n is odd is similar. From Lemma 2 and (4)

$$P'_n = \sum_{t=1}^{p_n-1} a^{q_n - [tq_{n-1}/p_{n-1}]} + a.$$

Set $t = ip_{n-1} + j$, $1 \leq j \leq p_{n-1}$. Then

$$1 \leq t \leq p_n - 1 = a_n p_{n-1} + p_{n-2} - 1$$

if and only if

$$0 \leq i \leq a_n - 1 \quad \text{and} \quad 1 \leq j \leq p_{n-1}$$

or

$$i = a_n \quad \text{and} \quad 1 \leq j \leq p_{n-2} - 1.$$

Thus

$$\begin{aligned}
P'_n &= \sum_{i,j} a^{q_n - i q_{n-1} - [j q_{n-1}/p_{n-1}]} + a \\
&= \sum_{j=1}^{p_{n-1}} a^{q_n - [j q_{n-1}/p_{n-1}]} \sum_{i=0}^{a_n-1} a^{-i q_{n-1}} \\
&\quad + \sum_{j=1}^{p_{n-2}-1} a^{q_n - a_n q_{n-1} - [j q_{n-1}/p_{n-1}]} + a \\
&= \frac{a^{-a_n q_{n-1}} - 1}{a^{-q_{n-1}} - 1} \sum_{j=1}^{p_{n-1}} a^{q_n - [j q_{n-1}/p_{n-1}]} + \sum_{j=1}^{p_{n-2}-1} a^{q_{n-2} - [j q_{n-1}/p_{n-1}]} + a \\
&= \frac{a^{q_n} - a^{q_{n-2}}}{a^{q_{n-1}} - 1} \sum_{j=1}^{p_{n-1}} a^{q_{n-1} - [j q_{n-1}/p_{n-1}]} \\
&\quad + \sum_{j=1}^{p_{n-2}-1} a^{q_{n-2} - [j q_{n-2}/p_{n-2}]} + a \quad (\text{by Lemma 2}) \\
&= t_n P'_{n-1} + P'_{n-2}
\end{aligned}$$

by induction. The proof is complete.

4. The transcendence of $S_a(\alpha)$. From Roth's theorem [1, p. 104] we have that if $|q\beta - p| < 1/q^{1.4}$ has an infinite number of integer solutions q, p , then β is transcendental. If A_n/B_n are the convergents of β then

$$|B_n \beta - A_n| < 1/B_{n+1} \leq 1/B_n^{1.4}$$

provided $\log B_{n+1}/\log B_n \geq 1.4$. For $B_n = Q_n$ above we have

$$\log Q_{n+1}/\log Q_n \sim q_{n+1}/q_n \quad (n \rightarrow \infty).$$

It is easily checked that $\max(q_{n+1}/q_n, q_n/q_{n-1}) \geq 1.5$. Thus

THEOREM. $S_a(\alpha)$ is transcendental for all irrationals α .

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