SHORTER NOTES

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FLAT OR OPEN IMPLIES GOING DOWN

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ABSTRACT. Let R, T be commutative rings with identity, and $f: R \rightarrow T$ a unital ring homomorphism. We give an elementary, unified proof of the fact that f has the going down property, if T is flat as an R -module or if the induced map $F: \text{Spec}(T) \rightarrow \text{Spec}(R)$ is open.

1. Introduction and the proof. Let R and T be commutative rings with identity and $f: R \rightarrow T$ a unital ring homomorphism. In [3, (5.D), Theorem 4], Matsumura gives a nice proof of the following fact: (a) If T is flat as an R -module, then f has the going down property; and in [1, I, Corollaire (3.9.4)], Grothendieck gives a straightforward proof of the following related result: (b) If the induced map $F: \text{Spec}(T) \rightarrow \text{Spec}(R)$ is an open map, then f satisfies the going down property. We give an elementary, unified proof of these statements. To simplify the notation, we may assume $R \subseteq T$.

PROOF OF (a) AND (b). Assume $R \subseteq T$ does not satisfy going down. We will show that the flatness or openness hypothesis is in contradiction with this assumption. Applying [2, Exercise 37, p. 44], there exist primes $P \in \text{Spec}(R)$ and $Q \in \text{Spec}(T)$ such that Q is minimal over PT , and

$$PT \cap (R \setminus P)(T \setminus Q) \neq \emptyset.$$

This produces an equation $\sum_{i=1}^n p_i t_i = rt$ for some $p_i \in P$, $t_i \in T$, $r \in R \setminus P$ and $t \in T \setminus Q$. To prove (a), write $\sum_{i=1}^{n+1} p_i t_i = 0$, with $p_{n+1} = -r$ and $t_{n+1} = t$. Thus, there exists an integer s and elements $r_{ij} \in R$ and $y_j \in T$ for $1 \leq j \leq s$, such that

$$(*) \quad \sum_{i=1}^{n+1} p_i r_{ij} = 0 \quad \text{for all } j,$$

and

$$(**) \quad t_i = \sum_{j=1}^s r_{ij} y_j \quad \text{for all } i$$

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[3, (3.A), Theorem 1]. Since $p_i \in P$ for $1 \leq i \leq n$ and $p_{n+1} \in R \setminus P$, (*) gives $r_{n+1,j} \in P$ for all j . Then (**), with $i = n + 1$, gives $t_{n+1} \in PT \subseteq Q$, contradicting $t \notin Q$.

To prove (b), let $X = \text{Spec}(T)$. As above, we have the equation $\sum_{i=1}^n p_i t_i = rt$. By assumption, $F(X_i)$ is open in $\text{Spec}(R)$, and thus $P \in F(X_i)$, since $Q \in X_i$ and $P \subseteq Q \cap R$. Hence, there is a $Q_1 \in X_i$ such that $Q_1 \cap R = P$. Thus, $r \in P$, a contradiction.

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