QUASI-COMPLEMENTORS AND COMPLEMENTORS ON CERTAIN BANACH ALGEBRAS

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ABSTRACT. We study quasi-complementors and complementors on annihilator Banach algebras. We give conditions for a quasi-complementor to be a complementor.

1. Introduction. Quasi-complemented algebras, which are generalizations of complemented algebras, were introduced and studied in [5]. Although these two classes of algebras are different in general, they have many similar properties as shown in [5]. In this paper, we continue the study of these algebras.

In §3, we study some properties of complementors on certain Banach algebras. Let A be a semisimple annihilator Banach algebra with a quasi-complementor q . Then we show that q is a complementor on A if and only if, for any orthogonal family of q -projections $\{f_\alpha\}$ and x in A , $\sum_\alpha f_\alpha x$ is summable in the norm of A and especially when $\{f_\alpha\}$ is a maximal family, $x = \sum_\alpha f_\alpha x$. Some particular cases of this result were known, though they appeared in different forms (e.g. see [4] and [7]).

In §4, we study continuous complementors. Let A be a B^* -algebra with a complementor p . Then we show that p is continuous if and only if the set E_p of all p -projections is a closed and bounded subset of A . Let A be a semisimple annihilator Banach algebra with a continuous complementor p , which has no minimal left ideals of dimension less than three. We show that A is a dense left ideal of some dual B^* -algebra B and $R^p = l(R)^* \cap A$ for all closed right ideals R of A .

2. Notation and preliminaries. For any subset S in an algebra A , let $l_A(S)$ and $r_A(S)$ denote the left and right annihilators of S in A , respectively. Let A be a Banach algebra. Then A is called an annihilator algebra, if for any closed left ideal J and for any closed right ideal R , we have $r_A(J) = (0)$ if and only if $J = A$ and $l_A(R) = (0)$ if and only if $R = A$. If $l_A(r_A(J)) = J$ and $r_A(l_A(R)) = R$, then A is called a dual algebra.

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Let A be a Banach algebra which is a subalgebra of a Banach algebra B . For each subset S of A , $\text{cl}(S)$ (resp. $\text{cl}_A(S)$) will denote the closure of S in B (resp. A). Also $l(S)$ and $r(S)$ (resp. $l_A(S)$ and $r_A(S)$) denote the left and right annihilators of S in B (resp. A). We write $\|\cdot\|$ for the norm on A and $|\cdot|$ for the norm on B .

Let A be a Banach algebra and let I_r be the set of all closed right ideals in A . Following [5], we shall say that A is a (right) quasi-complemented algebra if there exists a mapping $q: R \rightarrow R^q$ of I_r into itself having the following properties:

$$(2.1) \quad R \cap R^q = (0) \quad (R \in I_r);$$

$$(2.2) \quad (R^q)^q = R \quad (R \in I_r);$$

$$(2.3) \quad \text{if } R_1 \supset R_2, \text{ then } R_2^q \supset R_1^q \quad (R_1, R_2 \in I_r).$$

The mapping q is called a (right) quasi-complementor on A . $R + R^q$ is dense in A , $A^q = (0)$ and $(0)^q = A$ (see [5, p. 143]). Hence $R^q = (0)$ if and only if $R = A$.

A quasi-complemented algebra A is called a (right) complemented algebra if it satisfies

$$(2.4) \quad R + R^q = A \quad (R \in I_r).$$

In this case, the mapping q is called a (right) complementor on A (see [8, p. 651, Definition 1]).

In this paper, all algebras and linear spaces under consideration are over the complex field C . Definitions not explicitly given are taken from Rickart's book [7].

3. Quasi-complementors and complementors. Let A be a semisimple Banach algebra with a quasi-complementor q . In this section, we find conditions which imply that q is a complementor.

DEFINITION. A minimal idempotent f in a quasi-complemented Banach algebra A is called a q -projection if $(fA)^q = (1-f)A$. The set of all q -projections in A is denoted by E_q .

LEMMA 3.1. *Let A be a semisimple annihilator Banach algebra with a quasi-complementor q . Then*

- (i) *Every nonzero right ideal I of A contains a q -projection.*
- (ii) *Let R be a closed right ideal of A . If e and f are q -projections contained in R and R^q , respectively, then $ef = fe = 0$.*
- (iii) *If f_1 and f_2 are q -projections such that $f_1f_2 = 0$, then $f_2f_1 = 0$.*

PROOF. (i) Since A is an annihilator algebra, by [11, p. 37, Lemma 3.1] I contains a minimal right ideal J . It follows from the proof of [5, p. 149, Lemma 6.4] that J contains a (unique) q -projection which also belongs to I .

(ii) Since $eA \subset R$, $(eA)^q = (1-e)A \supset R^q$ and so $ef = 0$. Similarly $fe = 0$.

(iii) If $f_1 f_2 = 0$, then $f_2 \in (1 - f_1)A = (f_1 A)^q$. Thus it follows from (ii) that $f_2 f_1 = 0$. This completes the proof.

REMARK. Let A be a semisimple annihilator quasi-complemented Banach algebra. Then it follows from Zorn's Lemma and Lemma 3.1 that A contains a maximal orthogonal family $\{f_\alpha\}$ of q -projections; $A = \text{cl}_A(\sum_\alpha f_\alpha A)$.

If A is a semisimple complemented Banach algebra, then $x \in \text{cl}_A(xA)$ for all x in A (see [2, p. 39, Lemma 3]).

NOTATION. Let A be a quasi-complemented Banach algebra and R a closed right ideal of A . We denote by P_R the projection on R along R^q , i.e., $P_R(x + y) = x$ for all x in R and y in R^q . If $R + R^q = A$, then P_R is continuous.

THEOREM 3.2. *Let A be a semisimple annihilator Banach algebra with a quasi-complementor q . Then the following statements are equivalent:*

- (i) q is a complementor on A .
- (ii) *For any orthogonal family of q -projections $\{f_\alpha\}$ and x in A , $\sum_\alpha f_\alpha x$ is summable in the norm of A and especially when $\{f_\alpha\}$ is a maximal family, $x = \sum_\alpha f_\alpha x$.*

PROOF. Let f be a q -projection, I a closed right ideal and $f \in I_A(I)$. Then $I \subset (1 - f)A = (fA)^q$ and so $fA \subset I^q$. With this observation and Lemma 3.1, the proof of Theorem 3.2 now follows exactly like the proof of [4, p. 233, Theorem 3.8], with "hermitian idempotent" replaced by " q -projection" and " $I_A(I)^*$ " replaced by " I^q ".

Let $\{E_\lambda: \lambda \in \Lambda\}$ be the set of all finite sums $f_{\lambda_1} + \cdots + f_{\lambda_n}$. For each $\lambda \in \Lambda$, let P_λ be the projection on $E_\lambda A$ along $(E_\lambda A)^q$.

COROLLARY 3.3. *If q is a complementor on A , then $\{\|P_\lambda\|: \lambda \in \Lambda\}$ is bounded, where $\|P_\lambda\|$ denotes the operator bound of P_λ .*

PROOF. Clearly we can assume that $\{f_\alpha\}$ is a maximal family. Then by Theorem 3.2, $x = \sum_\alpha f_\alpha x$ for all x in A . Hence by [4, p. 231, Theorem 3.4], there exists a constant K (independent of x) such that

$$\|P_\lambda x\| = \|E_\lambda x\| \leq K\|x\|.$$

Therefore $\{\|P_\lambda\|: \lambda \in \Lambda\}$ is bounded.

THEOREM 3.4. *Let A be a semisimple annihilator Banach algebra which is a dense subalgebra of a B^* -algebra B such that $|\cdot|$ majorizes $\|\cdot\|$ on A and $x \in \text{cl}_A(xA)$ for all x in A . Then A is a left ideal of B if and only if the mapping $p: R \rightarrow I(R)^* \cap A$ is a complementor on the closed right ideals R of A .*

PROOF. It is well known that B is a dual B^* -algebra and so the mapping $J \rightarrow I(J)^*$ is a complementor on the closed right ideals J of B (see [8, p. 652]). Therefore p is a quasi-complementor on A by the proof of [5, p. 148, Theorem 5.3]. Clearly the set of all hermitian minimal idempotents in B is the set of all p -projections. Theorem 3.4 now follows from Theorem 3.2 and the proof of [10, p. 422, Theorem 5.2].

For commutative algebras, we have the following stronger version.

COROLLARY 3.5. *Let A be a semisimple commutative complemented Banach algebra with carrier space X_A . Then A is a dual algebra which is a dense ideal of $C_0(X_A)$, the algebra of all complex-valued continuous functions on X_A which vanish at infinity.*

PROOF. Since the socle of A is dense in A [5, p. 143, Lemma 3.3], for every proper closed ideal R of A , there exists some minimal idempotent e such that $e \notin R$. Then $eR = (0)$ and so A is an annihilator algebra. Since $x \in \text{cl}_A(xA)$ for all x in A , A is a dual algebra by [7, p. 106, Theorem (2.8.29)]. It is easy to see that A is a dense subalgebra of $B = C_0(X_A)$ and $\|\cdot\|$ majorizes $|\cdot|$ on A . By [8, p. 652, Lemma 1],

$$R^p = I_A(R) = I(R) \cap A$$

for all closed ideals R of A . Hence by Theorem 3.4, A is an ideal of B .

4. Continuous complementors. Let A be a semisimple annihilator Banach algebra with a complementor p and M_A the set of all minimal right ideals of A . For each $R \in M_A$, by Lemma 3.1, $R = fA$ for some p -projection f in A . Therefore $R + R^p = fA + (1 - f)A$. Let P_R be the projection on R along R^p . Then P_R is continuous. The following definition is given in [1, p. 387, Definition].

DEFINITION. Suppose $a_n \in A$ with $a_n A \in M_A$ ($n = 0, 1, 2, \dots$). A complementor p on A is said to be continuous if whenever a_n converges to a_0 , then $P_{a_n A}$ converges to $P_{a_0 A}$ uniformly on bounded subsets of any minimal left ideal of A .

In this section, let A be a semisimple annihilator complemented Banach algebra and $\{I_\lambda : \lambda \in \Lambda\}$ the family of all minimal closed two-sided ideals of A . Define p_λ by $R^{p_\lambda} = R^p \cap I_\lambda$ for all closed right ideals R of I_λ . Then A is the direct topological sum of $\{I_\lambda : \lambda \in \Lambda\}$ and p_λ is a quasi-complementor on I_λ . Let H_λ be a minimal left ideal of I_λ . Then H_λ is a Hilbert space under some equivalent inner product norm. Let B_λ be the algebra of all completely continuous linear operators on H_λ . Then I_λ is a dense subalgebra of B_λ such that $\|\cdot\|$ majorizes $|\cdot|_\lambda$ on I_λ , where $|\cdot|_\lambda$ is the norm on B_λ (see [8]).

If u and v are elements of a Hilbert space H , $u \otimes v$ will denote the operator on H defined by the relation $(u \otimes v)(h) = (h, v)u$ for all h in H .

LEMMA 4.1. *Let A be a simple finite dimensional B^* -algebra with a complementor p and E_p the set of all p -projections in A . Then p is continuous if and only if E_p is a closed and bounded subset of A .*

PROOF. Let H be a minimal left ideal of A and E the set of all hermitian minimal idempotents in A . Then H is a Hilbert space and A can be taken as the B^* -algebra of all linear operators on H . Let Q be the p -derived mapping of p (see [3, p. 463, Definition 3.7]). By [1, p. 388, Theorem 2.4], Q is continuous if and only if p is continuous.

Let $S = \{u \in H: \|u\| = 1\}$; then $u \rightarrow u \otimes u$ is a continuous mapping of S onto E . If p is continuous, then $u \rightarrow Q(u \otimes u)$ is a continuous mapping of S onto E_p . Since H is finite dimensional, S is compact and so is E_p . Thus E_p is closed and bounded. The "if" part of the lemma follows from [9, p. 257, Lemma 2].

The following result is similar to [10, p. 257, Theorem 3]. We do not assume any dimension restriction on the minimal left ideals of A here.

THEOREM 4.2. *Let A be a B^* -algebra with a complementor p . Then the following statements are equivalent:*

- (i) *The complementor p on A is continuous.*
- (ii) *The set E_p of all p -projections is a closed and bounded subset of A .*

PROOF. Let E_p^λ be the set of all p -projections in I_λ ; clearly $E_p^\lambda = E_p \cap I_\lambda$.

(i) $\Rightarrow$ (ii). Suppose that p is continuous. We show that each E_p^λ is closed and bounded. In fact, this is true if H_λ is finite dimensional by Lemma 4.1. If H_λ is infinite dimensional, then by [1, p. 395, Theorem 4.6] and [9, p. 257, Theorem 3], E_p^λ is closed and bounded. Now it is easy to verify that E_p is closed (see the proof of [9, p. 257, Theorem 3]). It remains to show that E_p is bounded. Suppose this is not so. Then there exists a sequence $f_n \in E_p^{\lambda_n}$ such that $\{\|f_n\|\}$ is unbounded. Since $f_m f_n \in I_{\lambda_m} \cap I_{\lambda_n} = (0)$ ($m \neq n$), $\{f_n\}$ is an orthogonal family. Therefore by Corollary 3.3, $\{P_{f_n A}\}$ is bounded. Since B is a B^* -algebra, by [9, p. 259, Theorem 4] $\{f_n\}$ is bounded, which is a contradiction. Hence E_p is bounded and so (ii) follows.

(ii) $\Rightarrow$ (i). Since $E_p^\lambda = E_p \cap I_\lambda$ ($\lambda \in \Lambda$), by Lemma 4.1 and [3, p. 471, Theorem 6.8] and [1, p. 387, Theorem 2.2], p is continuous. This completes the proof.

REMARK. "Complementor" cannot be replaced by "quasi-complementor" in Theorem 4.2. In fact, let B and p be given as in [1, p. 396, Example 1]. Then p is a continuous quasi-complementor on B , but not a complementor. It is easy to show that E_p is unbounded. Since each P_n is a continuous complementor on A_n , E_p is closed.

THEOREM 4.3. *Let A be a semisimple annihilator Banach algebra with a continuous complementor p . Suppose that A has no minimal left ideals of dimension less than three. Then A is a dense left ideal of some dual B^* -algebra B and $R^p = I(R)^* \cap A$ for all closed right ideals R of A .*

PROOF. Let p_λ , H_λ , I_λ and B_λ be defined as before and $|\cdot|_\lambda$ the norm on B_λ . By [1, p. 390, Theorem 3.2], p_λ induces a complementor q_λ on B_λ and by [1, p. 391, Theorem 3.3], q_λ has the form $J_\lambda^{q_\lambda} = I(J_\lambda)^*$ for all closed right ideals J_λ in B_λ . By [1, p. 393, Lemma 4.3], there exists a constant M such that

$$(4.1) \quad \|h\| \leq |h|_\lambda \leq M\|h\| \quad (h \in H_\lambda, \lambda \in \Lambda).$$

Let B be the $B^*(\infty)$ -sum of $\{B_\lambda: \lambda \in \Lambda\}$. Then B is a dual B^* -algebra. Since $A = \text{cl}_A(\sum_\lambda I_\lambda)$, it follows easily from (4.1) that A can be continuously

embedded in B and is a dense subalgebra of B . Let H be the Hilbert space given in [1, p. 394, Theorem 4.4]. Then by [1, p. 394, Theorem 4.4], we have

$$R^p = \{a \in A: aH \perp RH\} = \{a \in A: a^*R = (0)\}.$$

Therefore $R^p = l(R)^* \cap A$ and so by Theorem 3.4, A is a left ideal of B . This completes the proof.

REMARK 1. Theorem 4.3 shows that there is essentially one type of continuous complementor on A .

REMARK 2. As shown by an example in [1, p. 396], the dimension restriction in Theorem 4.3 cannot be removed.

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