NONSTANDARD MEASURE THEORY-HAUSDORFF MEASURE

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ABSTRACT. In this paper it is shown that the Hausdorff measures λ^t for $t \in [0, \infty)$ can be simultaneously represented as $\ast$ finite counting measures in an appropriate nonstandard model. That is, the following theorem is proved. **THEOREM.** *Suppose X is a metric space and ν is an infinite positive $\ast$ integer. Then there is a $\ast$ finite set G such that for every standard $t \in [0, \infty)$ and every λ^t -integrable Borel function, $f: X \rightarrow \mathbb{R}$,*

$$\int f d\lambda^t = \text{St}\left(\frac{1}{\nu^t} \sum_{x \in G} \ast f(x)\right).$$

Suppose $F = \{x_1, x_2, \dots, x_k\}$ is a finite subset of the unit interval $I = [0, 1]$. Then one can define a counting measure μ_F for arbitrary subsets A of I by $\mu_F(A) = k^{-1} \|F \cap A\|$, where $\|F \cap A\|$ denotes the cardinality of $F \cap A$. Such a measure is, of course, countably additive. If $\ast\mathcal{N}$ is an appropriate nonstandard model of the reals it contains nonstandard $\ast$ integers ν greater than any standard integer, and $\ast$ finite sets $F = \{x_1, x_2, \dots, x_\nu\}$ which exhibit all the formal properties of finite sets. In particular, for any standard set $A \subset I$, A has a natural extension $\ast A$ in $\ast\mathcal{N}$ and $\ast\|F \cap \ast A\|$ makes sense. Thus, one can define a counting measure

$$\bar{\mu}_F(A) = \nu^{-1} \|F \cap \ast A\|.$$

$\bar{\mu}_F(A)$ in general is a nonstandard real number in $\ast[0, 1]$. Thus, there is a unique standard real number, denoted $\text{St}(\bar{\mu}_F(A))$, which is infinitely close to $\bar{\mu}_F(A)$. We define a real-valued measure μ_F by

$$\mu_F(A) = \text{St}(\bar{\mu}_F(A)) = \text{St}(\nu^{-1} \|F \cap \ast A\|).$$

μ_F is finitely additive, but not necessarily countably additive.

In [2] we showed that with some care a $\ast$ finite set F can be chosen such that:

- (i) if A is Lebesgue measurable then $\mu_F(A)$ is the Lebesgue measure of A ;
- (ii) F contains every standard real number.

Property (i) implies for any bounded measurable function f , $\int f dm = \text{St}(\nu^{-1} \sum_{x \in F} \ast f(x))$. In [1] Bernstein showed that a $\ast$ finite set F can be found such that this property holds for any integrable function f . In [5], Ward Henson obtained this same result for an arbitrary probability measure μ

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replacing Lebesgue measure. Property (ii) may be regarded as saying that F represents Hausdorff zero measure, λ^0 , in the sense that $\lambda^0(A) = \|F \cap {}^*A\|$. Moreover, property (ii) implies that for any nonempty Lebesgue null set, A , $\bar{\mu}_F(A)$ is a nonzero infinitesimal. This raises the intriguing possibility that conditional probability even for Lebesgue null sets can be defined by $P(A|B) = \text{St}(\bar{\mu}_F(A \cap B)/\bar{\mu}_F(B))$. Of course this definition of $P(A|B)$ is always legal, but it is not clear whether it is "right". In particular, different * finite sets F satisfying (i) and (ii) may give far different values of $P(A|B)$. In [7] and [8] Parikh and Parnes obtained some results concerning conditional probability in this setting.

The purpose of the present paper is to show that a * finite set F can be chosen which simultaneously represents the Hausdorff measures λ^t , $0 \leq t < \infty$, as counting measures in the above sense. In particular, this means that if B has finite nonzero Hausdorff t -measure then $P(A|B) = \lambda^t(A \cap B)/\lambda^t(B)$ so that, for such sets, $P(A|B)$ is the "right" definition of conditional probability. The main theorem we prove here is the following.

THEOREM. *Suppose X is a metric space and ν is an infinite positive * integer. Then there is a * finite set G such that for every standard $t \in [0, \infty)$ and every standard λ^t -integrable Borel function, f ,*

$$\int f d\lambda^t = \text{St}\left(\frac{1}{\nu^t} \sum_{x \in G} {}^*f(x)\right).$$

I. Preliminaries. For convenience we recall some facts about Hausdorff measure (see [3], [4], [10], or [11]).

I.1 DEFINITION. Suppose X is a metric space with metric d . If $A \subseteq X$ let $\delta(A)$ denote the diameter of A ; let $\Pi(A)$ denote the set of countable partitions of A . If $\mathcal{Q} = \{A_1, A_2, \dots\} \in \Pi(A)$ then let $\delta(\mathcal{Q}) = \sup_n \delta(A_n)$. Finally let $\Pi_\epsilon(A)$ be the set of all countable partitions $\mathcal{Q}$ of A such that $\delta(\mathcal{Q}) \leq \epsilon$. Then, for each $t \geq 0$, Hausdorff t -measure, λ^t , is defined by

$$\lambda^t(A) = \lim_{\epsilon \rightarrow 0} \inf_{\mathcal{Q} \in \Pi_\epsilon(A)} \sum_{n=1}^{\infty} [\delta(A_n)]^t.$$

The following results regarding Hausdorff measure are well known.

I.2 RESULTS.

(i)

$$\begin{aligned} \lambda^0(A) &= \|A\| && \text{if } A \text{ is finite,} \\ &= +\infty && \text{otherwise.} \end{aligned}$$

(ii) For each $t \in [0, \infty)$, λ^t is a (countably additive) Borel measure.

(iii) If $\lambda^t(A)$ is σ -finite and $s > t$ then $\lambda^s(A) = 0$.

Hausdorff measure λ^t has strong geometric meaning. For example, if X is Euclidean k -space, R^k , and C is a rectifiable curve in R^k then $\lambda^1(C)$ is its arclength. More generally, if M is a smooth t -dimensional submanifold of R^k , where $t < k$ is an integer, then $\lambda^t(M)$ is the t -dimensional volume of M .

divided by a constant factor α , equal to the volume of a ball of unit diameter in R' . Some authors (e.g. [4]) include this constant factor in the definition of λ' .

This paper will presuppose familiarity with one of the basic books on Nonstandard Analysis (e.g. [6], [9], or [12]). $\mathfrak{M}$ will denote a higher order model including the real numbers R and a given metric space X . $^*\mathfrak{M}$ will denote a κ -saturated elementary extension of $\mathfrak{M}$ where κ is any cardinal greater than that of the universe of $\mathfrak{M}$. If P is an entity in $\mathfrak{M}$ the corresponding entity in $^*\mathfrak{M}$ will be denoted *P . In particular, $\mathcal{F}$ will denote the set of finite subsets of X and $^*\mathcal{F}$ the * finite subsets of *X in $^*\mathfrak{M}$, ω will denote the set $\{0, 1, 2, \dots\}$ of natural numbers in $\mathfrak{M}$ and $^*\omega$ its extension in $^*\mathfrak{M}$. The cardinality function $\|\cdot\|: \mathcal{F} \rightarrow \omega$ extends to $^*\|\cdot\|: ^*\mathcal{F} \rightarrow ^*\omega$ in $^*\mathfrak{M}$ where it enjoys the same formal properties as in $\mathfrak{M}$. By abuse of notation we will write $\|A\|$ for $A \in ^*\mathcal{F}$. We will use the usual notation, $\text{St}(t)$, for the standard part of a (nearstandard) point $t \in ^*R$, and $a \approx b$ for a infinitely close to b . If $t \in ^*R$ is positive infinite we adopt the convention $\text{St}(t) = +\infty$.

II. Proof of the main theorem. The idea of the proof is to first find a * finite set F which simultaneously represents λ' for all $t \in [0, \infty)$, and then to modify this set using techniques of Ward Henson [5] to handle $\int f d\lambda'$ for λ' -integrable but unbounded f . Theorem II.1 accomplishes the first step and part of the second.

II.1 THEOREM. *Suppose X is a metric space and ν is an infinite integer. Then there is a * finite set F such that:*

(i) *For every standard $t \in [0, \infty)$ and every standard Borel set B ,*

$$\text{St}(\|F \cap ^*B\|/\nu^t) = \lambda'(B).$$

(ii) *For every standard $s \in (0, \infty)$ and every standard $f: X \rightarrow \mathbf{R}$ such that $\lambda^s(\{x | f(x) \neq 0\}) = 0$,*

$$\frac{1}{\nu^s} \sum_{x \in F} ^*f(x) \approx 0.$$

PROOF. (i) Let $\mathcal{Q}$ be the set of all triples $\langle B, t, \varepsilon \rangle$ where B is a standard Borel set, $t \in [0, \infty)$ and $\varepsilon \in (0, \infty)$. For each $\langle B, t, \varepsilon \rangle \in \mathcal{Q}$ let $\Phi_{\langle B, t, \varepsilon \rangle}(F)$ be the sentence which says

F is a * finite subset of *X and

(a) if $\lambda'(B) = +\infty$ then $\|F \cap ^*B\|/\nu^t > 1/\varepsilon$,

(b) if $\lambda'(B)$ is finite then $|\lambda'(B) - \|F \cap ^*B\|/\nu^t| < \varepsilon$.

(ii) Let $\mathcal{B}$ be the set of all triples $\langle f, s, \delta \rangle$ where f is a standard nonnegative Borel function, $s \in (0, \infty)$ and $\delta \in (0, \infty)$, and $\lambda^s(\{x | f(x) \neq 0\}) = 0$. For each $\langle f, s, \delta \rangle \in \mathcal{B}$, let $\Psi_{\langle f, s, \delta \rangle}(F)$ be the sentence which says

$$F \text{ is a } ^*\text{finite subset of } ^*X \text{ and } \frac{1}{\nu^s} \sum_{x \in F} ^*f(x) < \delta.$$

Let

$$\mathfrak{S} = \{ \Phi_{\langle B, t, \varepsilon \rangle}(F) | \langle B, t, \varepsilon \rangle \in \mathcal{Q} \} \cup \{ \Psi_{\langle f, s, \delta \rangle}(F) | \langle f, s, \delta \rangle \in \mathfrak{B} \}.$$

We will show that for every finite subset

$$\{ \Phi_{\langle B_1, t_1, \varepsilon_1 \rangle}(F), \dots, \Phi_{\langle B_n, t_n, \varepsilon_n \rangle}(F), \Psi_{\langle f_1, s_1, \delta_1 \rangle}(F), \dots, \Psi_{\langle f_k, s_k, \delta_k \rangle}(F) \}$$

of $\mathfrak{S}$ there is a *finite set F in ${}^*\mathfrak{M}$ such that

$$\begin{aligned} {}^*\mathfrak{M} \models & \Phi_{\langle B_1, t_1, \varepsilon_1 \rangle}(F) \& \dots \& \Psi_{\langle B_n, t_n, \varepsilon_n \rangle}(F) \\ & + \Psi_{\langle f_1, s_1, \delta_1 \rangle}(F) \& \dots \& \Psi_{\langle f_k, s_k, \delta_k \rangle}(F). \end{aligned}$$

Hence, since ${}^*\mathfrak{M}$ is κ -saturated and the cardinality of $\mathfrak{S}$ is less than κ there is a *finite set F such that for every sentence χ in $\mathfrak{S}$, ${}^*\mathfrak{M} \models \chi(F)$. The sentences $\Phi_{\langle B, t, \varepsilon \rangle}(F)$ establish conclusion (i) while the sentences $\Psi_{\langle f, s, \delta \rangle}(F)$ establish conclusion (ii).

Let $\Pi(B_1, \dots, B_n) = \{C_1, \dots, C_m\}$ be the partition of X generated by $B_1, \dots, B_n$. That is, a typical C_i is of the form $C_i = E_1 \cap E_2 \cap \dots \cap E_n$ where each E_j is either B_j or $X \setminus B_j$. For each C_i we will find a *finite set $F_i \subseteq {}^*C_i$ such that:

(i) If $\lambda^b(C_i)$ is finite then $\text{St}(\|F_i\|/\nu^b) = \lambda^b(C_i)$, for $j = 1, 2, \dots, n$.

(ii) If $\lambda^b(C_i)$ is infinite then so is $\|F_i\|/\nu^b$, for $j = 1, 2, \dots, n$.

(iii) $(1/\nu^s) \sum_{x \in F_i} {}^*f_j(x) \approx 0$, for $j = 1, 2, \dots, k$.

Then $F = \bigcup F_i$ will be the desired *finite set. From now on we work with each C_i individually.

If for every t_j , $\lambda^b(C_i) = 0$, let $F_i = \emptyset$. Otherwise, let t be the greatest t_j such that $\lambda^b(C_i) \neq 0$.

Let $S_j = \{x | f_j(x) \neq 0\}$.

Notice for each $s_j \leq t$, $\lambda^t(S_j) = 0$. Let $C'_i = C_i \setminus (\bigcup_{s_j < t} S_j)$. Thus, $\lambda^t(C'_i) = \lambda^t(C_i)$, and if we can find $F_i \subseteq C'_i$ satisfying (i) and (ii), then (iii) will follow for $s_j \leq t$.

Next let $h(x) = \max\{f_j(x) | t < s_j\}$ and let $s = \min\{s_j | t < s_j\}$. Let $D_n = \{x \in C'_i | h(x) \leq n\}$. Then $C'_i = \bigcup D_n$.

Case (i): $t = 0$. Then $C'_i = C_i$ since no $s_j \leq 0$. If C_i is finite let $F_i = C_i$; then for any f_j , $\sum_{x \in F_i} {}^*f_j(x)$ is finite, so $(1/\nu^s) \sum_{x \in F_i} {}^*f_j(x) \approx 0$.

If C_i is infinite then for every infinite ρ , D_ρ is (externally) infinite. Pick an infinite ρ such that $\rho/\nu^s \approx 0$. Let F_i be any *finite set such that $F_i \subseteq D_\rho$, and $\|F_i\|$ is an infinite integer such that $\|F_i\|\rho/\nu^s \approx 0$. Then for any f_j , $(1/\nu^s) \sum_{x \in F_i} {}^*f_j(x) \approx 0$. Also, $\text{St}(\|F_i\|) = +\infty = \lambda^0(C_i)$, so F_i is the desired set.

Case (ii): $t \neq 0$. Then $\lambda^t(C'_i) = \lim_{n \rightarrow \infty} \lambda^t(D_n) \neq 0$. Hence there is an n such that $\lambda^t(D_n) \neq 0$. In particular, D_n is infinite. If $\lambda^t(C'_i)$ is finite choose $F_i \subseteq D_n$ so that $\|F_i\|/\nu^t \approx \lambda^t(C'_i)$. Now, for $s_j > t$, $(1/\nu^s) \sum_{x \in F_i} {}^*f_j(x) \approx 0$. If $\lambda^t(C'_i)$ is infinite choose $F_i \subseteq D_n$ such that $\|F_i\|/\nu^t$ is infinite and $n\|F_i\|/\nu^s$ is infinitesimal. Then, for $s_j > t$, since $s_j \geq s$, $(1/\nu^s) \sum_{x \in F_i} {}^*f_j(x) \approx 0$.

This completes the proof.

II.2 LEMMA. Suppose X is a metric space, v is an infinite integer and F is a $\ast$ finite subset of $\ast X$ satisfying the conclusions of Theorem II.1. Suppose $t \in [0, \infty)$ and f is a standard nonnegative λ^t -integrable function. Then there is a $\ast$ finite set $H_f \subseteq F$ satisfying the same conclusions such that

$$\int f d\lambda^t = \text{St} \left(\frac{1}{v^t} \sum_{x \in H_f} \ast f(x) \right).$$

PROOF. For every standard set A with $\lambda^t(A) = 0$, $(1/v^t) \sum_{x \in F \cap \ast A} \ast f(x) \approx 0$ since F satisfies conclusion (ii) of Theorem II.1. Therefore, by a straightforward saturation argument there is an internal set B such that

(i) for every standard set A with $\lambda^t(A) = 0$, $\ast A \subseteq B$, and

(ii) $(1/v^t) \sum_{x \in F \cap B} \ast f(x) \approx 0$.

We will choose H_f so that $F \cap B \subseteq H_f$. This insures that for any standard Borel set A such that $\lambda^t(A) = 0$, and any $s < t$, $\text{St}(\|H_f \cap \ast A\|/v^s) = \text{St}(\|F \cap \ast A\|/v^s)$.

For each $n \in \omega$, let $A_n = \{x | f(x) > n\}$ and $f_n(x) = \min(f(x), n)$. Since f is λ^t -integrable, $\lambda^t(A_1) < \infty$. Thus $\lim_{n \rightarrow \infty} \lambda^t(A_n) = 0$ and, by a straightforward argument, for every infinite $\rho \in \ast \omega \setminus \omega$, $\|A_\rho \cap F\|/v^t \approx 0$. By the Monotone Convergence Theorem $\lim_{n \rightarrow \infty} \int f_n d\lambda^t = \int f d\lambda^t$.

If, for some (standard) n , $\int f_n d\lambda^t = \int f d\lambda^t$, then $\lambda^t(A_n) = 0$ and $f = f_n + f|_{A_n}$. Then F itself satisfies the theorem since by conclusion (ii) of Theorem II.1, $(1/v^t) \sum_{x \in F} \ast (f|_{A_n})(x) \approx 0$.

Thus, we may assume for every standard $n \in \omega$, $\int f_n d\lambda^t < \int f d\lambda^t$. Therefore, since each f_n is bounded, $(1/v^t) \sum_{x \in F} \ast f_n(x) < \ast \int f d\lambda^t$. So there is an infinite $\rho \in \ast \omega \setminus \omega$ such that $(1/v^t) \sum_{x \in F} \ast f_\rho(x) < \ast \int f d\lambda^t$. Let $H_f = (F \setminus A_\rho) \cup (F \cap B)$. Since $(1/v^t) \sum_{x \in F \cap B} \ast f(x) \approx 0$ we have

$$\text{St} \left(\frac{1}{v^t} \sum_{x \in H_f} \ast f(x) \right) = \int f d\lambda^t.$$

Now, for $s \geq t$, since $\|A_\rho \cap F\|/v^t \approx 0$, conclusion (i) of Theorem II.1 is unaffected. Also, conclusion (ii) of Theorem II.1 is unaffected since $H_f \subseteq F$.

There is one odd case to be considered—if $s < t$, $\lambda^s(A)$ is infinite and $\lambda^t(A) \neq 0$. But then $\|H_f \cap \ast A\|/v^t$ is not infinitesimal so $\|H_f \cap \ast A\|/v^s$ is infinite. This completes the proof.

II.3 THEOREM. Suppose X is a metric space and v is an infinite, positive $\ast$ integer. Then there is a $\ast$ finite set G such that for every standard $t \in [0, \infty)$ and every standard λ^t -integrable function, f ,

$$\int f d\lambda^t = \text{St} \left(\frac{1}{v^t} \sum_{x \in G} \ast f(x) \right).$$

PROOF. Let $\mathcal{C}$ be the set of all triples $\langle f, t, \varepsilon \rangle$ where $t \in [0, \infty)$, f is a standard nonnegative λ^t -integrable function and $\varepsilon \in (0, \infty)$. For each $\langle f, t, \varepsilon \rangle \in \mathcal{C}$ let $\Phi_{\langle f, t, \varepsilon \rangle}(G)$ be the sentence, G is a $\ast$ finite subset of $\ast X$ and:

$$\left| \frac{1}{\nu^t} \sum_{x \in G} {}^*f(x) - {}^*\int f d\lambda^t \right| < \varepsilon.$$

All we need show is that the sentences $\{\Phi_{\langle f, t, \varepsilon \rangle}(G)\}_{\langle f, t, \varepsilon \rangle \in \mathcal{C}}$ are finitely satisfiable in ${}^*\mathcal{N}$. But, given $\langle f_1, t_1, \varepsilon \rangle, \dots, \langle f_n, t_n, \varepsilon_n \rangle$ let F be obtained from Theorem II.1 and $F_{f_1}, \dots, F_{f_n}$ from f by Lemma II.2. Then $G = F_{f_1} \cap \dots \cap F_{f_n}$ is the desired * finite set. This completes the proof.

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