ISOMETRIES OF QUASITRIANGULAR OPERATOR ALGEBRAS

ALAN HOPENWASSER AND JOAN PLASTIRAS¹

ABSTRACT. Let (P_n) be an increasing sequence of finite rank projections on a separable Hilbert space. Assume P_n converges strongly to the identity operator I . The quasitriangular operator algebra determined by (P_n) is defined to be the set of all bounded linear operators T for which

$$\lim_{n \rightarrow \infty} \|(I - P_n)TP_n\| = 0.$$

In this note we prove that two quasitriangular algebras are unitarily equivalent if, and only if, there exists a unital linear isometry mapping one algebra onto the other.

This note adds one further piece of information about isomorphisms of quasitriangular operator algebras to the facts already gathered in [7]. If $\mathcal{P} = \{P_n\}_{n=1}^{\infty}$ is a sequence of finite rank projections increasing to the identity and acting on a separable Hilbert space $\mathcal{H}$, then the quasitriangular algebra $\mathcal{QT}(\mathcal{P})$ is defined to be the set of all bounded operators T in $\mathcal{B}(\mathcal{H})$ such that $\|P_n^\perp TP_n\| \rightarrow 0$ as $n \rightarrow \infty$. Two quasitriangular algebras, $\mathcal{QT}(\mathcal{P})$ and $\mathcal{QT}(\mathcal{R})$ are said to be *isometric* if there exists a linear isometry ϕ mapping $\mathcal{QT}(\mathcal{P})$ onto $\mathcal{QT}(\mathcal{R})$ such that $\phi(I) = I$. In [7] it is shown that the following three conditions are equivalent:

- (a) $\mathcal{QT}(\mathcal{P})$ is algebraically isomorphic to $\mathcal{QT}(\mathcal{R})$;
- (b) there exist positive integers n_0 and m_0 such that $\dim(P_{n_0+k}) = \dim(R_{m_0+k})$ for all $k \geq 0$;
- (c) $\mathcal{QT}(\mathcal{P})$ is unitarily equivalent to $\mathcal{QT}(\mathcal{R})$.

Here, we add one further equivalent condition to the list:

- (d) $\mathcal{QT}(\mathcal{P})$ is isometric to $\mathcal{QT}(\mathcal{R})$.

First, we prove a lemma concerning extensions of isometries which may be of independent interest. $\mathcal{C}(\mathcal{H})$ denotes the set of compact operators.

LEMMA. Let $\psi_0: \mathcal{C}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ be a linear isometry with the property that, for each pair of vectors $x, y \in \mathcal{H}$,

$$\sup\{|\langle \psi_0(T)x, y \rangle| : T \in \mathcal{C}(\mathcal{H}) \text{ and } \|T\| = 1\} = \|x\| \|y\|.$$

Then there exists a unique extension of ψ_0 to a linear isometry, ψ , mapping $\mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{H})$.

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PROOF. For each pair of vectors $x, y \in \mathcal{H}$ define a linear functional $\phi_{x,y}^0$ on $\mathcal{C}(\mathcal{H})$ by $\phi_{x,y}^0(T) = \langle \psi_0(T)x, y \rangle$, for all $T \in \mathcal{C}(\mathcal{H})$. The hypothesis on ψ_0 asserts that $\|\phi_{x,y}^0\| = \|x\| \|y\|$. By the Hahn-Banach theorem, there exists an extension, $\phi_{x,y}$, of $\phi_{x,y}^0$ to $\mathfrak{B}(\mathcal{H})$ such that $\|\phi_{x,y}\| = \|x\| \|y\|$. By [2, Theorem 3], $\phi_{x,y}$ has a unique decomposition $\phi_{x,y} = \alpha + \beta$, where α is ultraweakly continuous; $\beta|_{\mathcal{C}(\mathcal{H})} = 0$; and $\|\phi_{x,y}\| = \|\alpha\| + \|\beta\|$. Since $\|\phi_{x,y}\| = \|\phi_{x,y}|_{\mathcal{C}(\mathcal{H})}\| = \|\alpha\|$, we obtain $\beta = 0$; i.e. $\phi_{x,y}$ is ultraweakly continuous. This already proves the uniqueness of extensions, for if ψ_1 and ψ_2 are both linear isometries which extend ψ_0 , then the linear functionals $A \rightarrow \langle \psi_1(A)x, y \rangle$ and $A \rightarrow \langle \psi_2(A)x, y \rangle$ are both ultraweakly continuous and agree on $\mathcal{C}(\mathcal{H})$. Since $\mathcal{C}(\mathcal{H})$ is ultraweakly dense in $\mathfrak{B}(\mathcal{H})$, $\langle \psi_1(A)x, y \rangle = \langle \psi_2(A)x, y \rangle$, for all $x, y \in \mathcal{H}$. Hence $\psi_1 = \psi_2$.

The ultraweak continuity of the $\phi_{x,y}$ implies that, for each $A \in \mathfrak{B}(\mathcal{H})$, $(x, y) \rightarrow \phi_{x,y}(A)$ is a bilinear form on $\mathcal{H}$ bounded by $\|A\|$. Consequently there exists an operator $\psi(A)$ in $\mathfrak{B}(\mathcal{H})$ such that $\phi_{x,y}(A) = \langle \psi(A)x, y \rangle$, for all $x, y \in \mathcal{H}$, and $\|\psi(A)\| \leq \|A\|$. It is routine to check that ψ is linear and that $\psi|_{\mathcal{C}(\mathcal{H})} = \psi_0$. To prove that ψ is an isometry, let E_n be a sequence of finite rank projections such that $E_n \uparrow I$ in the strong topology. Let $A \in \mathfrak{B}(\mathcal{H})$ and let $A_n = AE_n$, for each n . For each pair $x, y \in \mathcal{H}$,

$$\langle \psi(A)x, y \rangle = \phi_{x,y}(A) = \lim_n \phi_{x,y}(A_n) = \lim_n \langle \psi_0(A_n)x, y \rangle.$$

Hence, given $\varepsilon > 0$, there exist unit vectors x and y such that $|\langle \psi(A)x, y \rangle| > \|\psi_0(A_n)\| - \varepsilon = \|A_n\| - \varepsilon$. Thus, $\|\psi(A)\| \geq \|A_n\|$, for all n . Since $\|A_n\| \rightarrow \|A\|$, we have $\|\psi(A)\| = \|A\|$, for all $A \in \mathfrak{B}(\mathcal{H})$, and ψ is an isometry.

REMARK. If we drop the condition $\sup|\langle \psi_0(T)x, y \rangle| = \|x\| \|y\|$, then the uniqueness is no longer true. Indeed, let $\text{id}: \mathfrak{B}(\mathcal{H}) \rightarrow \mathfrak{B}(\mathcal{H})$ be the identity map and 0 be the zero map. Also, let ϕ be any linear map of $\mathfrak{B}(\mathcal{H})$ into itself such that $\|\phi\| \leq 1$ and $\phi|_{\mathcal{C}(\mathcal{H})} = 0$. Then after a suitable identification of $\mathcal{H} \oplus \mathcal{H}$ with $\mathcal{H}$, $\text{id} \oplus 0$ and $\text{id} \oplus \phi$ are two distinct linear isometries of $\mathfrak{B}(\mathcal{H})$ which agree on $\mathcal{C}(\mathcal{H})$. The proof of existence also breaks down, as the ultracontinuity of the $\phi_{x,y}$ depends on the fact that the functional achieves its norm on $\mathcal{C}(\mathcal{H})$.

Let $\mathcal{Q}(\mathcal{P})$ denote $\mathcal{Q}\mathcal{T}(\mathcal{P}) \cap \mathcal{Q}\mathcal{T}(\mathcal{P})^*$ and note that $\mathcal{Q}(\mathcal{P}) = \{T \mid \|P_n T - TP_n\| \rightarrow 0 \text{ as } n \rightarrow \infty\}$ [6, Proposition 16].

THEOREM. *The quasitriangular algebras $\mathcal{Q}\mathcal{T}(\mathcal{P})$ and $\mathcal{Q}\mathcal{T}(\mathcal{R})$ are unitarily equivalent if and only if they are isometric.*

PROOF. It is clear that if $\mathcal{Q}\mathcal{T}(\mathcal{P})$ is unitarily equivalent to $\mathcal{Q}\mathcal{T}(\mathcal{R})$ then they are isometric. For the converse, suppose that $\phi: \mathcal{Q}\mathcal{T}(\mathcal{P}) \rightarrow \mathcal{Q}\mathcal{T}(\mathcal{R})$ is a linear isometry of $\mathcal{Q}\mathcal{T}(\mathcal{P})$ onto $\mathcal{Q}\mathcal{T}(\mathcal{R})$ such that $\phi(I) = I$. By [4, Lemma 8], ϕ preserves adjoints on $\mathcal{Q}(\mathcal{P})$ and hence maps $\mathcal{Q}(\mathcal{P})$ onto $\mathcal{Q}(\mathcal{R})$. By Theorem 8 in [4] the restriction ϕ_0 of ϕ to $\mathcal{Q}(\mathcal{P})$ is a C^* -isomorphism (Jordan isomorphism). Because $\mathcal{Q}(\mathcal{P})$ and $\mathcal{Q}(\mathcal{R})$ are irreducible C^* -algebras, we may conclude that ϕ_0 is either a $*$ -isomorphism or a $*$ -anti-isomorphism (see [8, Theorem 6.4] or [5, Theorem 2.6]).

Actually, ϕ_0 must be a $*$ -isomorphism; the assumption that ϕ_0 is a $*$ -anti-isomorphism leads to a contradiction, as follows. Let $\mathcal{E} = \{e_1, e_2, \dots\}$ be an orthonormal basis of $\mathcal{H}$ such that each R_n is the projection on the span of $\{e_1, \dots, e_k\}$, where $k = \dim R_n$. Let $t: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ be the transpose with respect to $\mathcal{E}$: t is defined by the condition $\langle t(A)e_i, e_j \rangle = \langle Ae_j, e_i \rangle$, for all $A \in \mathcal{B}(\mathcal{H})$. Note that t is an anti-isomorphism and a linear isometry of $\mathcal{B}(\mathcal{H})$ onto itself. Observe, further, that $t(\mathcal{Q}(\mathcal{R})) = \mathcal{Q}(\mathcal{R})$. Indeed, by the choice of $\mathcal{E}$, $t(R_n) = R_n$, for all n . Then

$$\begin{aligned} \|AR_n - R_nA\| &= \|t(AR_n - R_nA)\| = \|t(R_n)t(A) - t(A)t(R_n)\| \\ &= \|R_nt(A) - t(A)R_n\|, \end{aligned}$$

for all n . Thus, $A \in \mathcal{Q}(\mathcal{R})$ if, and only if, $t(A) \in \mathcal{Q}(\mathcal{R})$. Now let $\psi = t \circ \phi$. So, ψ is a linear isometry of $\mathcal{Q}(\mathcal{P})$ into $\mathcal{B}(\mathcal{H})$ and $\psi_0 = t \circ \phi_0$ is a $*$ -isomorphism of $\mathcal{Q}(\mathcal{P})$ onto $\mathcal{Q}(\mathcal{R})$. Since $\mathcal{Q}(\mathcal{P})$ and $\mathcal{Q}(\mathcal{R})$ both contain $\mathcal{C}(\mathcal{H})$, ψ_0 is implemented by a unitary operator V . ([1, Corollary 3, p. 20, Theorem 1.3.4].) By the lemma above, ψ is also implemented by V ; hence the image of $\mathcal{Q}(\mathcal{P})$ under ψ is a quasitriangular algebra $\mathcal{Q}(\mathcal{S})$, where $\mathcal{S} = (S_n) = (VP_nV^{-1})$. Therefore, t maps $\mathcal{Q}(\mathcal{R})$ onto $\mathcal{Q}(\mathcal{S})$. Let U be the unilateral shift with respect to $\mathcal{E}$, viz. $Ue_n = e_{n+1}$, for all n . Then the backward shift, U^* , is easily seen to be an element of $\mathcal{Q}(\mathcal{R})$; hence $U = t(U^*) \in \mathcal{Q}(\mathcal{S})$. But this says that U is a quasitriangular operator and Halmos has proven the contrary in [3].

Therefore, we know that ϕ_0 is a $*$ -isomorphism of $\mathcal{Q}(\mathcal{P})$ onto $\mathcal{Q}(\mathcal{R})$. But then, as above, it follows that ϕ_0 is implemented by a unitary operator V , and, from the lemma, that ϕ is implemented by V . Thus $\mathcal{Q}(\mathcal{P})$ is unitarily equivalent to $\mathcal{Q}(\mathcal{R})$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALABAMA, UNIVERSITY, ALABAMA 35486

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PENNSYLVANIA, PHILADELPHIA, PENNSYLVANIA 19174