MORE ON CONNECTEDNESS IM KLEINEN AND LOCAL CONNECTEDNESS IN $C(X)$

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ABSTRACT. Let X be a compact connected metric space and 2^X ($C(X)$) denote the hyperspace of closed subsets (subcontinua) of X . Let $M \in C(X)$. If 2^X is connected im kleinen at M , then $C(X)$ is locally arcwise connected at M . A characterization of connectedness im kleinen in $C(X)$ is given. Indecomposability of X is related to an absence of local connectedness in 2^X and $C(X)$. An example is given of a continuum X and a subcontinuum M such that $C(X)$ is connected im kleinen at M but not locally connected at M .

A *continuum* will be a compact connected metric space and X will denote a continuum throughout. Each of 2^X and $C(X)$ is endowed with the finite (Vietoris) topology and, since X is a continuum, each of 2^X and $C(X)$ is also a continuum (see [4]).

In an earlier paper by the author [1], the properties of local connectedness and connectedness im kleinen were characterized at points in 2^X in terms of the neighborhood structure of the corresponding sets in X . In this paper the internal structure of $C(X)$ is investigated with respect to these properties.

For notational purposes, small letters will denote elements of X , capital letters will denote subsets of X and elements of 2^X , and script letters will denote subsets of 2^X . If $A \subset X$, then A^* (int A) (bd A) will denote the closure (interior) (boundary) of A in X .

Let $x \in X$. Then X is *connected im kleinen* at x if for each open set U containing x there is a component of U which contains x in its interior. X is *locally connected* (*locally arcwise connected*) at x if for each open set U containing x there is a connected open set (arcwise connected open set) V such that $x \in V \subset U$.

If $A_1, \dots, A_n$ are subsets of X , then $N(A_1, \dots, A_n) = \{B \in 2^X \mid \text{for each } i = 1, \dots, n, B \cap A_i \neq \emptyset, \text{ and } B \subset \bigcup_{i=1}^n A_i\}$. An *order arc* in 2^X ($C(X)$) is an arc which is also a chain with respect to the partial order on 2^X ($C(X)$) induced by set inclusion. If $A, B \in 2^X$ ($C(X)$), then there exists an order arc in 2^X ($C(X)$) from A to B if and only if $A \subset B$ and each component of B intersects A (Lemmas 2.3 and 2.6 of [3]).

We begin with a theorem which answers Question 1 of [1] in the affirma-

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tive, and which in fact provides a more general result.

THEOREM 1. *Let $M \in C(X)$. If 2^X is connected im kleinen at M , then $C(X)$ is locally arcwise connected at M .*

PROOF. Let $M \in C(X)$ and $N(U_1, \dots, U_n) \cap C(X)$ be a basic open set in $C(X)$ containing M . Let $N(V_1, \dots, V_m)$ be an open set such that $M \in N(V_1, \dots, V_m)$ and $N(V_1, \dots, V_m)^* \subset N(U_1, \dots, U_n)$. Let $\mathcal{C}$ be the component of $N(V_1, \dots, V_m)$ which contains M . Since 2^X is connected im kleinen at M , $\mathcal{C}$ contains M in its interior, so there exists an open set $N(W_1, \dots, W_p)$ such that $M \in N(W_1, \dots, W_p) \subset \mathcal{C} \subset N(V_1, \dots, V_m)$. Let $C = (\cup \mathcal{C})^*$. Then $C \in C(X)$ and $C \in N(V_1, \dots, V_m)^*$, so $C \in N(U_1, \dots, U_n)$. Let $U = \cup_{i=1}^n U_i$ and note that $M, C \in N(W_1, \dots, W_p, U) \subset N(U_1, \dots, U_n)$.

We will now show that $N(W_1, \dots, W_p, U) \cap C(X)$ is arcwise connected. Let $L_1, L_2 \in N(W_1, \dots, W_p, U) \cap C(X)$, $L_1 \neq L_2$. Since $\cup_{i=1}^p W_i \subset C$, it follows that $L_1 \cap C \neq \emptyset$ and $L_2 \cap C \neq \emptyset$. Hence $L_1 \cup L_2 \cup C \in N(W_1, \dots, W_p, U) \cup C(X)$. So there exist order arcs $\mathcal{E}_1$ and $\mathcal{E}_2$ from L_1 to $L_1 \cup L_2 \cup C$ and from L_2 to $L_1 \cup L_2 \cup C$. It follows that there is an arc in $\mathcal{E}_1 \cup \mathcal{E}_2$ from L_1 to L_2 , and it is clear that $\mathcal{E}_1 \cup \mathcal{E}_2 \subset N(W_1, \dots, W_p, U) \cap C(X)$.

THEOREM 2. *Let $M \in C(X)$. Then $C(X)$ is connected im kleinen at M if and only if for each open set U containing M , each sequence of continua contained in U which converges to M is eventually contained in the component of U to which M belongs.*

PROOF. Suppose $C(X)$ is connected im kleinen at M . Let U be an open set containing M and let D denote the component of U to which M belongs. Let $\{L_i\}_{i=1}^\infty$ be a sequence of continua contained in U such that $L_i \rightarrow M$. Let $\mathcal{C}$ denote the component of $N(U) \cap C(X)$ to which M belongs. Since $C(X)$ is connected im kleinen at M there exists an open set $N(V_1, \dots, V_n) \cap C(X)$ such that $M \in N(V_1, \dots, V_n) \cap C(X) \subset \mathcal{C}$. Since $L_i \rightarrow M$, there exists a positive integer i_0 such that $i \geq i_0$ implies $L_i \in N(V_1, \dots, V_n) \cap C(X)$. Since $\mathcal{C}$ is a connected subset of $C(X)$, $\cup \{C \in C(X) | C \in \mathcal{C}\}$ is a connected subset of X , and $\cup \{C \in C(X) | C \in \mathcal{C}\} \subset U$. Then $\cup_{i=i_0}^\infty L_i \subset \cup \{C \in C(X) | C \in \mathcal{C}\} \subset D$. So $\{L_i\}_{i=1}^\infty$ is eventually contained in D .

For the converse, suppose that $C(X)$ is not connected im kleinen at M . Then there exists an open set $\mathcal{U} \cap C(X)$ and a sequence $\{\mathcal{C}_i\}_{i=1}^\infty$ of distinct components of $(\mathcal{U} \cap C(X))^*$ converging to a limit continuum $\mathcal{C}$ which contains M and has no point in common with any of the continua $\mathcal{C}_i$ (Theorem 12.1 on p. 18 of [5]). Choose basic open sets $N(U_1, \dots, U_n)$ and $N(V_1, \dots, V_m)$ each containing M such that

$$\begin{aligned} (N(U_1, \dots, U_n) \cap C(X))^* &\subset N(U_1, \dots, U_n)^* \cap C(X) \\ &\subset N(V_1, \dots, V_m) \cap C(X) \subset \mathcal{U} \cap C(X). \end{aligned}$$

Since $\mathcal{C}_i \rightarrow \mathcal{C}$ and $M \in \mathcal{C}$, there exists a sequence $\{L_i\}_{i=1}^\infty$ such that $L_i \in \mathcal{C}_i$ and $L_i \rightarrow M$. Hence there exists a positive integer i_0 such that $i > i_0$ implies $L_i \in N(U_1, \dots, U_n) \cap C(X)$. Let $U = \bigcup_{i=1}^n U_i$ and let E denote the component of U to which M belongs. Then $\{L_i\}_{i=i_0}^\infty$ is a sequence of continua contained in U which converges to M . Suppose that for some $i > i_0$, $L_i \cap E \neq \emptyset$. Then $L_i \subset E \subset E^*$ and $M \subset E^*$. Now $E^* \in N(V_1, \dots, V_m) \cap C(X)$, so it follows that there exist order arcs in $N(V_1, \dots, V_m) \cap C(X)$ from L_i to E^* and from M to E^* . Hence L_i and M belong to the same component of $\mathcal{U} \cap C(X)$. But $M \in \mathcal{C}$ and $L_i \in \mathcal{C}_i$ and $\mathcal{C} \cap \mathcal{C}_i = \emptyset$. It follows that for each $i > i_0$, $L_i \cap E = \emptyset$. Hence $\{L_i\}_{i=i_0}^\infty$ is not eventually in E .

THEOREM 3. *Let $M \in C(X)$. If $\text{int } M \neq \emptyset$, then $C(X)$ is locally arcwise connected at M .*

PROOF. Let $M \in C(X)$ and let $N(U_1, \dots, U_n) \cap C(X)$ be a basic open set containing M . Let $x \in \text{int } M$ and let V be an open set such that $x \in V \subset \text{int } M$ and such that $V \subset \bigcap \{U_i | x \in U_i\}$. Then $M \in N(U_1, \dots, U_n, V) \subset N(U_1, \dots, U_n)$. Let $L_1, L_2 \in N(U_1, \dots, U_n, V) \cap C(X)$. Then $L_1 \cap V \neq \emptyset$ and $L_2 \cap V \neq \emptyset$, so $L_1 \cap M \neq \emptyset$ and $L_2 \cap M \neq \emptyset$. It follows that $L_1 \cup L_2 \cup M \in N(U_1, \dots, U_n, V) \cap C(X)$. Hence there exist order arcs $\mathcal{C}_1$ and $\mathcal{C}_2$ in $N(U_1, \dots, U_n, V) \cap C(X)$ from L_1 to $L_1 \cup L_2 \cup M$ and from L_2 to $L_1 \cup L_2 \cup M$. An argument similar to that used in Theorem 1 will establish that $N(U_1, \dots, U_n, V) \cap C(X)$ contains an arc joining L_1 and L_2 .

It follows from Theorem 2 of [1] and Theorem 3 that for any continuum X , each of 2^X and $C(X)$ is locally connected at X . The next two results relate indecomposability to an absence of connectedness im kleinen in 2^X and $C(X)$.

THEOREM 4. *If X is indecomposable, then X is the only point at which 2^X is connected im kleinen.*

PROOF. Suppose $A \in 2^X$, $A \neq X$, and 2^X is connected im kleinen at A . Let B be any component of A and let $y \in X - A$. For each $x \in B$ let U_x be an open set such that $y \notin U_x$. Let $\{U_{x_1}, \dots, U_{x_n}\}$ be a finite subset of $\{U_x | x \in B\}$ such that $B \subset \bigcup_{i=1}^n U_{x_i}$. Then $B \in N(U_{x_1}, \dots, U_{x_n})$. Let $N(V_1, \dots, V_m)$ be an open set such that

$$B \in N(V_1, \dots, V_m) \subset N(V_1, \dots, V_m)^* \subset N(U_{x_1}, \dots, U_{x_n}).$$

By Theorem 4 of [1], 2^X is connected im kleinen at B . Hence there exists an open set $N(W_1, \dots, W_p)$ and a connected set $\mathcal{C}$ such that $B \in N(W_1, \dots, W_p) \subset \mathcal{C} \subset N(V_1, \dots, V_m)$. Let $D = (\bigcup \{C \in 2^X | C \in \mathcal{C}\})^*$. Then $D \in N(U_{x_1}, \dots, U_{x_n}) \cap C(X)$, $y \notin D$, and $\bigcup_{i=1}^p W_i \subset D$, so D is a proper subcontinuum of X with interior. It follows that X is decomposable.

The example which follows shows that the converse of Theorem 4 is false.

EXAMPLE 1. In Diagram 1 below, the planar continuum X contains a copy

of the Cantor fan with vertex x_0 . The points x_{2n} (y_{2n}) lie on the line segment joining x_0 and p (q) and the points x_{2n+1} (y_{2n+1}) lie on the line segment joining x_1 and p (y_1 and q). The first coordinates of the points x_n (y_n) form a strictly increasing (strictly decreasing) sequence converging to the first coordinate of p (q). For each n a Cantor set is chosen between x_n and x_{n+2} (y_n and y_{n+2}) and a Cantor fan is constructed with vertex x_{n+1} (y_{n+1}).

DIAGRAM 1

X is certainly decomposable, but it follows from Theorem 1 of [1] that there is no proper subcontinuum of X at which 2^X is connected im kleinen. Hence by Theorem 4 of [1], X is the only point at which 2^X is connected im kleinen.

THEOREM 5. *If X is the only point at which $C(X)$ is connected im kleinen, then X is indecomposable.*

PROOF. If X is decomposable, then X has a proper subcontinuum M with interior, and by Theorem 3, $C(X)$ is connected im kleinen at M .

The next example shows that the converse of Theorem 5 is false.

DIAGRAM 2

EXAMPLE 2. The indecomposable continuum X in Diagram 2 is a slightly modified version of the indecomposable continuum commonly referred to as the “buckethandle.” A small arc $[p, q]$ has been inserted at the top so that the continuum maintains the essential flavor of the “buckethandle” but contains a triod.

Let $A = [r, s]$ and let $N(U_1, \dots, U_n) \cap C(X)$ be a basic open set as in the diagram on the right, with each U_i chosen as a spherical open set of diameter $< \frac{1}{2}$ the distance from $[r, s]$ to q . If $B \in N(U_1, \dots, U_n) \cap C(X)$, then there is a triod $C \in N(U_1, \dots, U_n) \cap C(X)$ which contains both A and B , and hence there is an arc in $N(U_1, \dots, U_n) \cap C(X)$ joining A and B . It follows that $C(X)$ is locally arcwise connected at the proper subcontinuum A .

It follows from Corollaries 1 and 4 of [1] and Theorem 1 that if $x \in X$, then $C(X)$ is connected im kleinen at $\{x\}$ if and only if $C(X)$ is locally connected at $\{x\}$. Theorem 3 shows that the notions of connectedness im kleinen and local connectedness agree at those points $M \in C(X)$ for which $\text{int } M \neq \emptyset$. So it might be conjectured that these notions agree for all points of $C(X)$. Our final example shows that this is not the case.

EXAMPLE 3. This example is a rather extreme generalization of the example from p. 113 of [2]. We start by embedding the example just cited in the first quadrant of the yz plane in E^3 . Let $a(i, j)$ denote the left-hand endpoint of the j th arc in the i th harmonic fan and p_i denote the vertex of the i th harmonic fan. The fans are chosen so that if $a(i, j) \neq a(m, n)$, then their z -coordinates $z(i, j)$ and $z(m, n)$ are different. Let M denote the line segment joining $(\frac{1}{4}, 0, 0)$ and $(-\frac{1}{4}, 0, 0)$ and let $p = (0, 0, 0)$ (see Diagram 3).

DIAGRAM 3

Next we attach "indented" continua at each $a(i, j)$ consisting of five straight line segments in the $z(i, j)$ plane which terminate at $(\frac{1}{4}, 0, z(i, j))$ and $(-\frac{1}{4}, 0, z(i, j))$. Let $C(i, j)$ denote the union of the indented continuum attached at $a(i, j)$ and the arc $[a(i, j), p_i]$ joining $a(i, j)$ and p_i . The design is such that $\lim_{j \rightarrow \infty} C(i, j) = M \cup [p, p_i]$ (see Diagram 4). Let $Y_1 = (\cup \{C(i, j) | i, j \in N\}) \cup [p, p_1] \cup M$.

DIAGRAM 4

We now indicate briefly why $C(Y_1)$ is locally arcwise connected at M . Let $N(U_1, \dots, U_n) \cap C(Y_1)$ be a basic open set containing M such that $\{U_1, \dots, U_n\}$ consists of a simple chain of open spheres of diameter $< \frac{1}{8}$ from $(\frac{1}{4}, 0, 0)$ to $(-\frac{1}{4}, 0, 0)$. Let $i_0 = \inf\{i \in N | p_i \in \cup_{j=1}^n U_j\}$. It is apparent that if $D \in N(U_1, \dots, U_n) \cap C(Y_1)$ and there does not exist an arc in $N(U_1, \dots, U_n) \cap C(Y_1)$ joining M and D , then for some j_1 ,

$$D \subset C(i_0 - 1, j_1).$$

Let $C(i_0, j_2) \in N(U_1, \dots, U_n) \cap C(Y_1)$. Choose "small" arcwise connected open sets $N(V_1, \dots, V_m) \cap C(Y_1)$ and $N(W_1, \dots, W_p) \cap C(Y_1)$ containing D and $C(i_0, j_2)$, respectively, and let O be a "small" open set containing $[a(i_0 - 1, j_1), p_{i_0-1}] \cup [p_{i_0-1}, p_{i_0}]$. Then

$$\left[N \left[V_1, \dots, V_m, O \cup \left(\bigcup_{i=1}^p W_i \right) \right] \cup N \left[W_1, \dots, W_p, O \cup \left(\bigcup_{i=1}^m V_i \right) \right] \right] \cap C(Y_1)$$

is an arcwise connected open set containing D and $C(i_0, j_2)$. It follows that we can "enlarge" $N(U_1, \dots, U_n) \cap C(Y_1)$ to an arcwise connected open set $\mathcal{V}$ such that for some $p^1 \in (p_{i_0-2}, p_{i_0-1})$,

$$\cup \mathcal{V} \subset M \cup [p, p^1] \cup (\cup \{C(i, j) | i \geq i_0 - 1, j \in N\}).$$

The purpose of the inductive process which follows is to introduce enough points of nonconnectedness im kleinen to prevent the resulting hyperspace from being locally connected at M .

Next we construct a sequence $C(2, 1, j^1)$ of "indented" continua so that $\lim_{j^1 \rightarrow \infty} C(2, 1, j^1) = C(2, 1) \cup [p_1, p_2]$ and such that $(\cup \{C(2, 1, j^1) | j^1 \in N\}) \cap Y_1 = \{p_1\}$. Further, if $j_1^1 \neq j_2^1$, we require that $C(2, 1, j_1^1) \cap C(2, 1, j_2^1) = \{p_1\}$. Inductively, for each positive integer j we construct a sequence $C(2, j, j^1)$ so that $\lim_{j^1 \rightarrow \infty} C(2, j, j^1) = C(2, j) \cup [p_1, p_2]$. The sequences are chosen so that $(\cup \{C(2, j, j^1) | j, j^1 \in N\}) \cap Y_1 = \{p_1\}$ and such that if $j_1 \neq j_2$ or $j_1^1 \neq j_2^1$, then $C(2, j_1, j_1^1) \cap C(2, j_2, j_2^1) = \{p_1\}$ (see Diagram 5). Let $Y_2 = Y_1 \cup (\cup \{C(2, j, j^1) | j, j^1 \in N\})$.

DIAGRAM 5

We now proceed to define Y_i inductively. For each positive integer j we construct a sequence $C(i, j, j^1)$ such that

$$\lim_{j^1 \rightarrow \infty} C(i, j, j^1) = C(i, j) \cup [p_{i-1}, p_i],$$

such that

$$(\cup \{C(i, j, j^1) | j, j^1 \in N\}) \cap Y_{i-1} = \{p_{i-1}\},$$

and such that if $j_1 \neq j_2$ or $j_1^1 \neq j_2^1$, then $C(i, j_1, j_1^1) \cap C(i, j_2, j_2^1) = \{p_{i-1}\}$.

Inductively, for each $k = 1, \dots, i - 2$ and for each $j, j^1, \dots, j^k$, we construct a sequence $C(i, j, j^1, \dots, j^k, j^{k+1})$ so that

$$\begin{aligned} \lim_{j^{k+1} \rightarrow \infty} C(i, j, j^1, \dots, j^k, j^{k+1}) \\ = C(i, j, j^1, \dots, j^k) \cup [p_{i-(k+1)}, p_{i-k}] \end{aligned}$$

and such that

$$\begin{aligned} \left(\bigcup \{ C(i, j, j^1, \dots, j^k, j^{k+1}) \mid j, j^1, \dots, j^k, j^{k+1} \in N \} \right) \\ \cap \left(Y_{i-1} \cup \left(\bigcup \{ C(i, j, j^1, \dots, j^k) \mid j, j^1, \dots, j^k \in N \} \right) \right) = \{ p_{i-(k+1)} \}. \end{aligned}$$

Further we require that if $j_1 \neq j_2$ or if for at least one $i \in \{1, \dots, k + 1\}$, $j_1^i \neq j_2^i$, then

$$C(i, j_1, j_1^1, \dots, j_1^k, j_1^{k+1}) \cap C(i, j_2, j_2^1, \dots, j_2^k, j_2^{k+1}) = \{ p_{i-(k+1)} \}$$

(see Diagram 6). Now we let

$$\begin{aligned} Y_i &= Y_{i-1} \cup \left(\bigcup \{ C(i, j, j^1) \mid j, j^1 \in N \} \right) \\ &\quad \cup \left(\bigcup \{ C(i, j, j^1, j^2) \mid j, j^1, j^2 \in N \} \right) \cup \dots \\ &\quad \cup \left(\bigcup \{ C(i, j, j^1, \dots, j^{i-1}) \mid j, j^1, \dots, j^{i-1} \in N \} \right). \end{aligned}$$

Finally we let $X = \bigcup_{i=1}^{\infty} Y_i$.

DIAGRAM 6

It follows from Theorem 2 that $C(X)$ is connected im kleinen at M . Suppose $N(U_1, \dots, U_n) \cap C(X)$ is the type of open set described after the first stage of the construction. If $\mathcal{V}$ is a connected open set and $M \in \mathcal{V} \subset N(U_1, \dots, U_n) \cap C(X)$, then for some $i \in N$, $M \cup [p, p_i] \in \mathcal{V}$. So there exists a sequence $\{B_j\}_{j=1}^{\infty}$ such that $B_j \rightarrow M \cup [p, p_i]$ and such that for each $j \in N$, $B_j \subset C(i - 1, j)$. Now any connected subset of $N(U_1, \dots, U_n) \cap C(X)$ containing B_j and M must contain a continuum $D(i - 1, j) \subset C(i - 1, j)$ such that $p_{i-1} \in D(i - 1, j)$. Thus for some j_1 there exists a sequence $\{B(j_1, j^1)\}_{j^1=1}^{\infty}$ such that $B(j_1, j^1) \rightarrow D(i - 1, j_1)$, and such that for each $j^1 \in N$, $B(j_1, j^1) \subset C(i - 1, j_1, j^1)$. It follows that $\mathcal{V}$ must contain a continuum $D(i - 1, j_1, j^1) \subset C(i - 1, j_1, j^1)$ such that $p_{i-2} \in D(i - 1, j_1, j^1)$. Continuing this process, after $i - 1$ steps we obtain $p_1 \in \bigcup \mathcal{V}$. Hence $C(X)$ is not locally connected at M .

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