CONVOLUTION POWERS OF SINGULAR-SYMMETRIC MEASURES. II

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ABSTRACT. Let G be an infinite compact abelian group such that its dual group contains an infinite independent subset. $\mathcal{L}(G)$ denotes the sum of all radicals of group algebras contained in the measure algebra on G . Then, for a positive integer k , there is a measure μ on G such that μ^n is singular-symmetric for $1 < n < k$ and $\mu^n \in \mathcal{L}(G)$ for $n > k$.

Let G be an infinite compact abelian group and $\hat{G}$ its dual group. A subset E of G is said to be independent if for every choice of distinct points $x_1, \dots, x_i$ of E and integers $n_1, \dots, n_i$, either $n_1x_1 = n_2x_2 = \dots = n_ix_i = 0$ or $n_1x_1 + n_2x_2 + \dots + n_ix_i \neq 0$. In this paper, we assume that $\hat{G}$ contains an infinite independent set. Let $M(G)$ be the measure algebra on G and $L^1(G)$ the group algebra on G . We denote by $\text{Rad } L^1(G)$ the radical of $L^1(G)$ in $M(G)$. We put $\mathcal{L}(G) = \sum_{\tau} \text{Rad } L^1(G_{\tau})$, where τ runs through over L.C.A. group topologies on G which are stronger than the original one. We denote by Δ the maximal ideal space of $M(G)$. For $\mu \in M(G)$, we put $\mu^*(E) = \overline{\mu(-E)}$ for every Borel subset E of G . We denote by $\mathfrak{M}$ the set of all symmetric measures of $M(G)$, that is, $\mathfrak{M} = \{\mu \in M(G); \tilde{\mu}^*(f) = \overline{\tilde{\mu}(f)} \text{ for every } f \in \Delta\}$, where $\tilde{\mu}$ is the Gel'fand transform of μ . Then we have $\mathcal{L}(G) \subset \mathfrak{M}$. A measure $\mu \in \mathfrak{M}$ is called singular-symmetric if μ is singular with $\mathcal{L}(G)$ ($\mu \perp \mathcal{L}(G)$).

In [1] and [2], the author showed that there exists a singular-symmetric measure on $\bar{R}$ such that $\mu * \mu \in \mathcal{L}(\bar{R})$, where $\bar{R}$ is the Bohr compactification of the real line R . In [5], Shimizu showed that there exists a singular-symmetric measure on $H = \coprod H_n$, where H_n is an infinite compact abelian group ($n = 1, 2, \dots$), using essentially the same method as in [1]. In the previous paper [3], the author showed that there exists a singular-symmetric measure μ on $\bar{R}$ such that μ^n is singular-symmetric for every positive integer n , where $\mu^n = \mu^{n-1} * \mu$ and $\mu^1 = \mu$ ($n \geq 2$).

In this paper, we show the following. The essential idea of the proof is found in [1].

THEOREM. *Let G be a compact abelian group whose dual group $\hat{G}$ contains an infinite independent set. Then, for each positive integer k , there exists a*

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singular-symmetric measure μ on G such that

- (a) μ^n is singular-symmetric if $1 \leq n \leq k$,
- (b) $\mu^n \in \mathcal{L}(G)$ if $n > k$.

The proof of the Theorem proceeds in steps.

(I) We fix a positive integer k . Let E be an infinite countable independent subset of $\hat{G}$.

LEMMA. *There exists a family of infinite subsets $\{E_{n,i}\}_{0 \leq n < \infty; 1 \leq i \leq 2^n}$ of E such that:*

- (1) $E_{n,i}$ is an infinite subset of E for $0 \leq n < \infty$ and $1 \leq i \leq 2^n$;
- (2) $E_{n,i} \subsetneq E_{n+1,2i-1}$ and $E_{n,i} \subsetneq E_{n+1,2i}$ for $0 \leq n < \infty$ and $1 \leq i \leq 2^n$;
- (3) $E_n \cap E_{n+1,2i-1} = E_n \cap E_{n+1,2i} = E_{n,i}$, where $E_n = \bigcup \{E_{n,i}; 1 \leq i \leq 2^n\}$, for $0 \leq n < \infty$ and $1 \leq i \leq 2^n$;
- (4) for $s > n > k$ and $1 \leq i_1 < i_2 < \cdots < i_n \leq 2^s$, we have

$$E_{s-1} \cap \left(\bigcap_{q=1}^n E_{s,i_q} \right) = \bigcap_{q=1}^n E_{s,i_q};$$

- (5) for $s > k \geq n_1 \geq n_2$, $1 \leq i_1 < i_2 < \cdots < i_{n_1} \leq 2^s$ and $1 \leq j_1 < j_2 < \cdots < j_{n_2} \leq 2^s$, we have

$$\left(\bigcap_{q=1}^{n_2} E_{s,j_q} \setminus E_{s-1} \right) \setminus \left(\bigcap_{q=1}^{n_1} E_{s,i_q} \setminus E_{s-1} \right)$$

is an infinite set if $\{j_1, j_2, \dots, j_{n_2}\} \subsetneq \{i_1, i_2, \dots, i_{n_1}\}$,

$$\left(\bigcap_{q=1}^{n_2} E_{s,j_q} \setminus E_{s-1} \right) \setminus \left(\bigcap_{q=1}^{n_1} E_{s,i_q} \setminus E_{s-1} \right)$$

and

$$\left(\bigcap_{q=1}^{n_1} E_{s,i_q} \setminus E_{s-1} \right) \setminus \left(\bigcap_{q=1}^{n_2} E_{s,j_q} \setminus E_{s-1} \right)$$

are infinite sets if $\{j_1, j_2, \dots, j_{n_2}\} \not\subset \{i_1, i_2, \dots, i_{n_1}\}$.

PROOF. This Lemma is an elementary set-theoretic fact about infinite sets, so we do not give the detailed proof. In the next step, we construct a measure μ on G using the properties of $\{E_{n,i}\}_{n,i}$ of the Lemma, and in Steps (III) and (IV) we show that μ satisfies conditions (a) and (b) of the Theorem. Property (4) is used to show (a) and property (5) is used to show (b). In [1], we constructed a measure μ on G such that $\mu \perp \mathcal{L}(G)$ and $\mu * \mu \in \mathcal{L}(G)$, using a certain family of subsets $\{E_\alpha\}_{\alpha \in \Lambda}$ of E . Lemma 1 of [1] is the case $k = 1$ of this Lemma. In [3], we constructed a measure μ on G such that μ^n is singular-symmetric for every positive integer n , using a certain family of subsets $\{E_{n,i}\}_{n,i}$ of E . But the properties of $\{E_{n,i}\}$ in [3] differ extremely from those in this Lemma.

(II) Let $H_{n,i}$ be the subgroup of $\hat{G}$ generated by $E_{n,i}$, and $H_n = \bigcup \{H_{n,i}; 1 \leq i \leq 2^n\}$. Then we have:

(6) for $0 \leq n < \infty$ and $1 \leq i \leq 2^n$, $H_n \cap H_{n+1, 2i-1} = H_n \cap H_{n+1, 2i} = H_{n,i}$ by (3);

(7) for $s \geq n > k$ and $1 \leq i_1 < i_2 < \dots < i_n \leq 2^s$, we have

$$\bigcap_{q=1}^n \left(\bigcup_{t \geq s} \left(\bigcup \{H_{t,j}; 2^{t-s}(i_q - 1) + 1 \leq j \leq 2^{t-s}i_q\} \right) \right) \\ = \bigcap_{q=1}^n H_{s,i_q} \quad \text{by (4).}$$

Let $G_{n,i} = H_{n,i}^\perp$ be the annihilator of $H_{n,i}$ in G . Then $\{G_{n,i}\}_{n,i}$ are compact subgroups of G and $(\bigcap_{t=1}^q H_{n,i_t})^\perp = \bigoplus_{1 \leq t \leq q} G_{n,i_t}$, where

$$\bigoplus_{1 \leq t \leq q} G_{n,i_t} = \{x_1 + x_2 + \dots + x_q; x_t \in G_{n,i_t} (t = 1, 2, \dots, q)\}.$$

We put

$$(8) \quad \mu_n = \sum_{i=1}^{2^n} \left(\frac{1}{2} \right)^n m(G_{n,i}),$$

where $m(G_{n,i})$ is the normalized Haar measure on $G_{n,i}$; then $\mu_n \geq 0$ and $\|\mu_n\| = 1$. For $\gamma \in H_s \setminus H_{s-1}$, we put $\beta_\gamma = \text{card}\{i; 1 \leq i \leq 2^s, \gamma \in H_{s,i}\}$. Then we have $\hat{\mu}_s(\gamma) = \beta_\gamma (\frac{1}{2})^s$ by (8), where $\hat{\mu}_s$ is the Fourier-Stieltjes transform of μ_s . By (6), we have $\text{card}\{i; 1 \leq i \leq 2^n, \gamma \in H_{n,i}\} = 2^{n-s}\beta_\gamma$ for $n > s$. Then $\hat{\mu}_n(\gamma) = (\frac{1}{2})^n 2^{n-s}\beta_\gamma = (\frac{1}{2})^s \beta_\gamma = \hat{\mu}_s(\gamma)$ for every $n > s$. If $\gamma \notin \bigcup_{s=0}^\infty H_s$, then $\hat{\mu}_n(\gamma) = 0$ for $n = 0, 1, 2, \dots$. These imply that $\{\mu_n\}_{n=0}^\infty$ has only one weak*-cluster point μ in $M(G)$, and $\mu \geq 0$, $\|\mu\| = 1$.

(III) We show that μ satisfies condition (a) of our Theorem. We fix a positive integer n such that $n > k$ and prove (a). For $s \geq k$ and $1 \leq i \leq 2^s$, we put $\lambda_{s,i,t} = (\frac{1}{2})^t \Sigma \{m(G_{t,j}); 2^{t-s}(i-1) + 1 \leq j \leq 2^{t-s}i\}$ for $t \geq s$. Then $\{\lambda_{s,i,t}\}_{t=s}^\infty$ has only one weak*-cluster point $\lambda_{s,i}$ in $M(G)$ in the same way as the previous part, and we have:

$$(9) \quad \mu = \sum_{i=1}^{2^s} \lambda_{s,i} \text{ for } s = 1, 2, \dots;$$

$$(10) \quad \lambda_{s,i} \text{ is concentrated on } G_{s,i} \text{ for } s \geq k \text{ and } 1 \leq i \leq 2^s;$$

$$(11) \quad \hat{\lambda}_{s,i}(\gamma) = (\frac{1}{2})^s \text{ for } \gamma \in H_{s,i};$$

$$(12) \quad \hat{\lambda}_{s,i}(\gamma) \neq 0 \text{ for } \gamma \in \bigcup_{t \geq s} (\bigcup \{H_{t,j}; 2^{t-s}(i-1) + 1 \leq j \leq 2^{t-s}i\}), \hat{\lambda}_{s,i}(\gamma) = 0 \text{ for other } \gamma \text{ in } \hat{G}.$$

For $s \geq n > k$ and $1 \leq i_1 < i_2 < \dots < i_n \leq 2^s$, we have $\hat{\lambda}_{s,i_q}(\gamma) = 0$ for $\gamma \notin \bigcap_{q=1}^n (\bigcup_{t \geq s} (\bigcup \{H_{t,j}; 2^{t-s}(i_q-1) + 1 \leq j \leq 2^{t-s}i_q\}))$ by (12). By (7), we have $\hat{\lambda}_{s,i_q}(\gamma) = 0$ for $\gamma \notin \bigcap_{q=1}^n H_{s,i_q}$. By (11), we have $\prod_{q=1}^n \hat{\lambda}_{s,i_q} = (\frac{1}{2})^{ns}$ on $\bigcap_{q=1}^n H_{s,i_q}$, so that

$$(13) \quad \bigstar_{q=1}^n \lambda_{s,i_q} = \left(\frac{1}{2} \right)^{ns} m \left(\bigoplus_{1 \leq q \leq n} G_{s,i_q} \right) \in \mathcal{L}(G).$$

We put $\mu^n = (\sum_{i=1}^{2^s} \lambda_{s,i})^n = P_s + P'_s$, where $P_s = \Sigma \{n! \bigstar_{q=1}^n \lambda_{s,i_q}; 1 \leq i_1$

$\{ \dots < i_n \leq 2^s \}$ and P'_s includes squares and higher powers of the $\lambda_{s,i}$. Here we note the following fact:

$$(14) \quad \left\| \sum \left\{ n! \cdot \prod_{q=1}^n \lambda_{s,i_q}; 1 \leq i_1 < i_2 < \dots < i_n \leq 2^s \right\} \right\| = \binom{2^s}{n} n! \left(\frac{1}{2} \right)^{ns} \\ \rightarrow 1 \quad (s \rightarrow \infty) \quad \text{for every positive integer } n.$$

Since $1 = \|\mu^n\| = \|P_s\| + \|P'_s\|$ and $\|P_s\| \rightarrow 1$ ($s \rightarrow \infty$) by (14), we have $\|P'_s\| \rightarrow 0$. Therefore $P_s \rightarrow \mu^n$ ($s \rightarrow \infty$) in norm. Since $\mathcal{L}(G)$ is norm closed and $P_s \in \mathcal{L}(G)$ ($s > n > k$) by (13), we have $\mu^n \in \mathcal{L}(G)$ ($n > k$). This completes the proof of (a).

(IV) We fix a positive integer n such that $1 \leq n \leq k$, and prove (b). Suppose that $\mu^n \notin \mathcal{L}(G)$. Then there is a L.C.A. group topology τ on G which is stronger than the original one such that $\mu^n \notin \text{Rad } L^1(G_\tau)$. Also, this implies $\mu^{k+1} \notin \text{Rad } L^1(G_\tau)$. We put $\mu^{k+1} = \eta_1 + \eta_2$ and $\varepsilon = \|\eta_1\| > 0$, where η_1 is the part of μ^{k+1} which is contained in $\text{Rad } L^1(G_\tau)$ and $\eta_1 \perp \eta_2$. By (14), there is a sufficient large positive integer s such that $s > k$ and

$$\left\| (k+1)! \sum \left\{ \prod_{q=1}^{k+1} \lambda_{s,i_q}; 1 \leq i_1 < \dots < i_{k+1} \leq 2^s \right\} \right\| > 1 - \varepsilon.$$

Then there exists $(j_1, j_2, \dots, j_{k+1})$ such that $1 \leq j_1 < \dots < j_{k+1} \leq 2^s$ and $\prod_{q=1}^{k+1} \lambda_{s,j_q} \notin \text{Rad } L^1(G_\tau)$. Then by (13), τ is the topology on G that makes $\bigoplus_{1 \leq q \leq k+1} G_{s,j_q}$ an open compact subgroup. We put

$$Q = \{(i_1, i_2, \dots, i_n); 1 \leq i_1 \leq i_2 \leq \dots \leq i_n \leq 2^s\},$$

$$Q_1 = \{(i_1, \dots, i_n) \in Q; \{i_1, \dots, i_n\} \subset \{j_1, j_2, \dots, j_{k+1}\}\},$$

$$Q_2 = \{(i_1, \dots, i_n) \in Q; \{i_1, \dots, i_n\} \not\subset \{j_1, j_2, \dots, j_{k+1}\}\}.$$

If $(i_1, \dots, i_n) \in Q_1$, then $\bigcap_{q=1}^{k+1} H_{s,j_q} \subsetneq \bigcap_{q=1}^n H_{s,i_q}$, and this implies that

$$\bigoplus_{1 \leq q \leq k+1} G_{s,j_q} = \left(\bigcap_{q=1}^{k+1} H_{s,j_q} \right)^\perp \supsetneq \left(\bigcap_{q=1}^n H_{s,i_q} \right)^\perp = \bigoplus_{1 \leq q \leq n} G_{s,i_q},$$

and $\bigoplus_{1 \leq q \leq k+1} G_{s,j_q} / \bigoplus_{1 \leq q \leq n} G_{s,i_q}$ is an infinite compact group by (4) and (5). Then we have $m_\tau(\bigoplus_{1 \leq q \leq n} G_{s,i_q}) = 0$, where m_τ is a Haar measure on G_τ . If $(i_1, \dots, i_n) \in Q_2$, then $\bigcap_{q=1}^{k+1} H_{s,j_q} \not\supset \bigcap_{q=1}^n H_{s,i_q}$, and this implies also $m_\tau(\bigoplus_{1 \leq q \leq n} G_{s,i_q}) = 0$ by (4) and (5), so that we have $m_\tau(\bigoplus_{1 \leq q \leq n} G_{s,i_q}) = 0$ for $(i_1, \dots, i_n) \in Q$. Since

$$\mu^n = \left(\sum_{i=1}^{2^s} \lambda_{s,i} \right)^n = \sum \left\{ b(i_1 \cdots i_n) \prod_{q=1}^n \lambda_{s,i_q}; (i_1, \dots, i_n) \in Q \right\},$$

where $b(i_1, \dots, i_n)$ is a positive integer, the support of μ^n is contained in

$\cup \{ \bigoplus_{1 \leq q \leq n} G_{s,i_q}; (i_1, \dots, i_n) \in Q \}$ by (10). By the above facts, we have $m_\tau(\cup \{ \bigoplus_{1 \leq q \leq n} G_{s,i_q}; (i_1, \dots, i_n) \in Q \}) = 0$. This implies that $\mu^n \perp L^1(G_\tau)$. Let $(i_1, \dots, i_n) \in Q$ and $p > 1$. Then (10) implies that $(\ast_{q=1}^n \lambda_{s,i_q})^p$ is supported in $\bigoplus_{1 \leq q \leq n} G_{s,i_q}$. Therefore $(\ast_{q=1}^n \lambda_{s,i_q})^p \perp L^1(G_\tau)$. This shows that $(\ast_{q=1}^n \lambda_{s,i_q}) \perp \text{Rad } L^1(G_\tau)$ [6]. Then we have $\mu^n \perp \text{Rad } L^1(G_\tau)$. This is a contradiction. This completes the proof of (b).

REMARK 1. For $\lambda \in M(G)$, we denote by $\sigma(\lambda)$ the spectrum of λ , that is, $\sigma(\lambda) = \{ \tilde{\lambda}(f); f \in \Delta \}$. Let μ be the measure that is constructed in our Theorem. Then $\sigma(\mu)$ is a countable set and there are no nonzero elements in $\sigma(\mu)$ that are cluster points of $\sigma(\mu)$.

REMARK 2. For fixed positive integers $k < q$, there are $\mu_1, \mu_2, \dots, \mu_q \in M(G)$ such that $(\mu_i)^s$ is singular-symmetric for $1 \leq s \leq k$ and $(\mu_i)^s \in \mathcal{L}(G)$ for $s > k$ ($i = 1, 2, \dots, q$), and $\ast_{i=1}^q \mu_i$ is singular-symmetric. Because, for an infinite independent subset E of $\hat{G}$, let E' be an infinite subset of E such that $E \setminus E'$ is an infinite subset. We construct $\mu_1 \in M(G)$ such that $(\mu_1)^s \perp \mathcal{L}(G)$ for $1 \leq s \leq k$, $(\mu_1)^s \in \mathcal{L}(G)$ for $s > k$, and $\hat{\mu}_1 = 0$ on $\hat{G} \setminus H'$, where H' is the subgroup of $\hat{G}$ generated by E' . Also, we construct $\mu_2, \dots, \mu_q \in M(G)$ such that $\|\mu_i\| = 1$, $(\mu_i)^s \perp \mathcal{L}(G)$ for $1 \leq s \leq k$ and $(\mu_i)^s \in \mathcal{L}(G)$ for $s > k$ and $\hat{\mu}_i = 1$ on H' ($i = 2, 3, \dots, q$). Then $\ast_{i=1}^q \mu_i = \mu_1 \perp \mathcal{L}(G)$.

REMARK 3. For the circle group, there cannot exist any measure such as we have constructed in our Theorem. (This is an elementary argument.) Whether there exist any singular-symmetric measures on the circle (necessarily with all their powers singular) remains an open question.

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