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ON A THEOREM OF POSNER

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ABSTRACT. Posner [1] has proved the following theorem: *Let R be a prime ring*[1] *and d a derivation of R such that, for all $a \in R$, $ada - daa \in Z$ (centre of R). Then, if d is not a zero derivation, R is commutative.*

Two proofs of this theorem are known–one by Posner himself and another by Ram Awtar [2]. It is natural to expect that the proof would be short and simple in the case when R is of characteristic 2, but, by chance, these proofs are somewhat contrary to this expectation. Although, in this case, the condition $ada - daa \in Z$ is the same as $d(a^2) \in Z$, which implies $d(ab + ba) \in Z$ for all $a, b \in R$.

The object of this paper is to give a very simple, short and direct proof of this theorem in the case when R is of characteristic 2, then to generalize this technique to prove a lemma which is similar to the result that, if $d(ab - ba) \in Z$, when R is not necessarily of characteristic 2, then either d is zero or R is commutative, and lastly to prove a generalized form of the main theorem, i.e. Theorem 1 of Awtar [3], which follows as a consequence of the lemma.

PROOF OF POSNER'S THEOREM, IN THE CASE WHEN R IS OF CHARACTERISTIC 2. The condition $ada - daa \in Z$ is the same as $d(a^2) \in Z$, which gives $d(ab + ba) \in Z$ if we replace a by $(a + b)$. Commuting $d(ab + ba)$ with a we have

$$\text{(i)} \qquad ad(ab + ba) = d(ab + ba)a.$$

Replacing b by ba in (i) and using (i), we have

$$\text{(ii)} \qquad a(ab + ba)da = (ab + ba)daa.$$

Replacing b by da in (ii), since $ada - daa \in Z$, and R is of characteristic 2, we have

$$(ada - daa)^2 = 0.$$

Since R is prime, this implies $ada - daa = 0$. Consequently, we have

$$d(ab + ba) = 0.$$

Replacing b by ba, we have

$$(ab + ba)da = 0.$$

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[1]All rings considered in this paper are associative. For definitions see [4].

Replacing b by bc, $c \in R$, in the last result and using the last result, since R is prime, it follows that either d is zero or R is commutative.

LEMMA. *Let R be a prime ring of characteristic not 2. Let S be the subgroup of the additive group of R generated by all $AB - BA$ and $(AB - BA)A$, where $A = xy - yx$, $B = ab - ba$, and a, b, x, y run through R. If d is a derivation of R such that $dV \in Z$ (centre of R) for all $V \in S$, then d is zero or R is commutative.*

PROOF. Since $d(A_1B_1 - B_1A_1) \in Z$, choosing $A_1 = V = AB - BA$ or $(AB - BA)A$, $B_1 = Vx - xV$, $x \in R$, we have $d[V(Vx - xV) - (Vx - xV)V] \in Z$. Since $dV \in Z$, it follows easily that $p^2(dx) \in Z$, where $p(x) = Vx - xV$. Replacing x by xV in this result, we have

(I) $$p^2(dx)V + p^2(x)\, dV \in Z.$$

Putting dx for x in (I), and using the results $dV \in Z, p^2(dx) \in Z$, we have

(II) $$p^2(d^2x)V \in Z.$$

Since $p^2(d^2x) \in Z$ and R is prime, either $p^2(d^2x) = 0$ or $V \in Z$. If $p^2(d^2x) = 0$, replacing x by xV in this result, we have

$$p^2(d^2xV + 2dxdV + xd^2V) = 0,$$

i.e.

(III) $$2p^2(dx)\, dV + p^2(x)\, d^2V = 0,$$

because $xdV - dVx = 0$, and so $d(xdV - dVx) = 0$ or $xd^2V - d^2Vx = 0$, i.e. $d^2V \in Z$. Since $dV \in Z$ and $p^2(dx) \in Z$, (III) implies $p^2(x)\, d^2V \in Z$. Since R is prime, either $d^2V = 0$ or $p^2(x) \in Z$. If $d^2V = 0$, since R is not of characteristic 2, by (III), we have

$$p^2(dx)\, dV = 0.$$

Since $dV \in Z$ and R is prime, we have $dV = 0$ or $p^2(dx) = 0$. If $p^2(dx) = 0$, by (I), we have $p^2(x)\, dV \in Z$, so either $dV = 0$ or $p^2(x) \in Z$. If $p^2(x) \in Z$, replacing x by xV, we have $p^2(x)V \in Z$. Therefore, either $p^2(x) = 0$ or $V \in Z$. If $p^2(x) = 0$, replacing x by $xy, y \in R$, it follows easily that $V \in Z$, and since $V = AB - BA$ or $(AB - BA)A$, we have $AB - BA = 0$, i.e. $V = 0$. Hence in each case $dV = 0$.

Since $dV = 0$, we have $d(AB - BA) = 0$ and $d[(AB - BA)A] = 0$. Therefore, we have

(IV) $$(AB - BA)\, dA = 0.$$

Replacing B by V_1 in (IV), where $V_1 = AB_1 - B_1A$, and using (IV), we have

(V) $$(AB_1 - B_1A)AdA = 0.$$

Again, replacing B_1 by $AB_2 - B_2A$ in (V), and using (V), we have

$$(AB_2 - B_2A)A^2dA = 0.$$

Therefore, it follows that

(VI) $$VdA = 0 \quad \text{and} \quad VAdA = 0,$$

for all $V \in S$. Replacing B by $Vx - xV$ in (IV), $V \in S$, $x \in R$, and using (VI), we have

(VII) $$(AVx - VxA)\,dA = 0.$$

Replacing x by xB_1 in (VII), where $B_1 = zy - yz, y, z \in R$, we have

$$\left[(AVx - VxA)B_1 + Vx(AB_1 - B_1A)\right] dA = 0.$$

By (IV), with B_1 for B, the second term on the left-hand side vanishes, so we have

(VIII) $$(AVx - VxA)(zy - yz)\,dA = 0.$$

Replacing $AVx - VxA$ by H and y by Hy in (VIII), we have

$$H\left[(zH - Hz)y + H(zy - yz)\right] dA = 0.$$

By (VIII), the second term on the left-hand side vanishes, and since R is prime, we have

$$H(zH - Hz) = 0 \quad \text{or} \quad dA = 0.$$

Therefore, either $H \in Z$ or $dA = 0$. If $H \in Z$, by (VII), it follows that either $H = 0$ or $dA = 0$. If $H = 0$, i.e. if $AVx - VxA = 0$, replacing x by SAt, S, $t \in R$, and using $H = 0$, with x replaced by S, we have

$$VSA(At - tA) = 0.$$

Therefore, either $V = 0$ or $A \in Z$. If $V = 0$, then A commutes with B, i.e. A commutes with $(xy - yx)$, $x, y \in R$. Replacing y by yx, it follows that A commutes with $(xy - yx)x$, so we have

$$(xy - yx)(xA - Ax) = 0.$$

Therefore, replacing y by yz, $z \in R$, it follows that $A \in Z$, i.e. $xy - yx \in Z$, so $(xy - yx)x \in Z$, which implies that R is commutative. If $A \in Z$, then A commutes with B, and it follows, as before, that R is commutative. If $dA = 0$, we have $d(xy - yx) = 0$, $x, y \in R$. Replacing y by yx, we have $(xy - yx)\,dx = 0$. Replacing y by yz, $z \in R$, it follows that either d is zero or R is commutative. This completes the proof of the lemma.

THEOREM. *Let $T(S)$ be a subgroup of the additive group of R, where R and S are the same as before, such that $V \in T(S)$ and $Vt - tV \in T(S)$, for every V and t, $V \in S$, $t \in R$. Let us suppose that R is not of characteristic 2 or 3, and let d be a nonzero derivation of R, with $UdU - dUU \in Z$ (centre of R) for all $U \in T(S)$. Then R is commutative.*

PROOF. Since by hypothesis $V \in T(S)$ and $Vr - rV \in T(S)$, for all V and r, $V \in S$, $r \in R$, replacing u by V in Lemma 2 of Ram Awtar [2], it follows that $[[dr, V], V] \in Z$, where $[x, y] = xy - yx$, $x, y \in R$. Now, proceeding on the same lines as in Posner [1] (cf. equations (16) to (27)), we have $[V, dV] = 0$, for all $V \in S$. Putting rV for r in $p^2(dr) \in Z$, where $p(r) = Vr - rV$, and using

$$p^2(xy) = p^2(x)y + 2p(x)p(y) + xp^2(y), \qquad x, y \in R,$$

we have

$$p^2(dr)V + p^2(r)\, dV \in Z.$$

Commuting this with V and using $[V, dV] = 0, p^2(dr) \in Z$, we have

$$\left[V, p^2(r)\right] dV = 0,$$

i.e. $p^3(r)\, dV = 0$. Putting yr for $r, y \in R$, and using

$$p^3(yr) = p^3(y)r + 3p^2(y)p(r) + 3p(y)p^2(r) + yp^3(r),$$

we have

$$\left\{p^3(y)r + 3p^2(y)p(r) + 3p(y)p^2(r)\right\}dV = 0.$$

Putting dy for y and $p(r)$ for r, and using $p^2(dy) \in Z$, $p^3(dy) = 0$, we have either $p^2(dy) = 0$ or $p^2(r)\, dV = 0$. If $p^2(r)\, dV = 0$, then, since $p^2(dy)V + p^2(r)\, dV \in Z$ and $p^2(dy) \in Z$, it follows that $p^2(dy) = 0$. Replacing y by rV, we have $p^2(r)\, dV = 0$. Replacing r by $yp(r)$, we have $p^2(y)p(r)\, dV = 0$, and replacing r by $p(y)r$ in $p^2(r)\, dV = 0$, we have either $dV = 0$ or $p^3(y) = 0$. If $p^3(y) = 0$, replacing y by yr, we have $p^2(y)p(r) + p(y)p^2(r) = 0$. Replacing y by dy, we have $p(dy)p^2(r) = 0$, and replacing y by $p(y)$ in $p^2(y)p(r) + p(y)p^2(r) = 0$, we have $p^2(y)p^2(r) = 0$. Replacing y by ydx, $x \in R$, we have $p^2(y)\, dxp^2(r) = 0$. Replacing x by $(xp^2(t))a$, $t \in R$, $a \in R$, we have either $p^2(r) = 0$ and, consequently, $V \in Z$, or $d(p^2(t)) = 0$, which reduces to $I_{dV}I_V(t) = 0$, where I_a denotes inner derivation by a, because $p^2(dt) = 0$ and $VdV = dVV$. Consequently, by Posner [1, Theorem 1], it follows that $V \in Z$ or $dV \in Z$. If $V \in Z$, we have $Vr - rV = 0, r \in R$. Therefore, $d(Vr - rV) = 0$, and we have $dV \in Z$, i.e. in each case $dV \in Z$ for all $V \in S$. Since $p^2(r)\, dV = 0$, it follows that either $dV = 0$ or $V \in Z$, for all $V \in S$. Hence, by the lemma, it follows that R is commutative.

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