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IDEAL BOUNDARIES OF A RIEMANN SURFACE FOR THE EQUATION $\Delta u = Pu$

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ABSTRACT. For a nonnegative density P on a hyperbolic Riemann surfaces R, let Δ^P be the subset of the Royden harmonic boundary consisting of the nondensity points of P. This ideal boundary, as well as the P-harmonic boundary δ_P of the P-compactification of R, have been employed in the study of energy-finite solutions of $\Delta u = Pu$ on R. We show that Δ^P is homeomorphic to $\delta_P - \{s_P\}$, where s_P is the P-singular point. It follows that δ_P fails to characterize the space $PBE(R)$ in the sense that it is possible for δ_P to be homeomorphic to δ_Q, but $PBE(R)$ is not canonically isomorphic to $QBE(R)$.

1. In order to study the space $PBE(R)$, or $PE(R)$, for a density $P \geqslant 0$ on a hyperbolic Riemann surface R, two ideal boundaries have been especially suited to the task. One is the subset Δ^P of the Royden harmonic boundary Δ, consisting of the nondensity points of P, and the other, δ_P, is the P-harmonic boundary of the P-compactification of R. The former was introduced in [1], and the latter in [6]. An interesting feature of δ_P is the existence of the P-singular point s_P, which in a sense corresponds to $\Delta \setminus \Delta^P$.

The question naturally arises as to what is the relation between Δ^P and δ_P, and in this note we prove that Δ^P and $\delta_P - \{s_P\}$ are homeomorphic.

In [2], it was shown that Δ^P does not characterize $PBE(R)$ in the sense that there exist densities P and Q on R with $\Delta^P = \Delta^Q$, but $PBE(R)$ is not canonically isomorphic to $QBE(R)$. Here, a canonical isomorphism is a vector space isomorphism ψ: $PBE(R) \to QBE(R)$ such that for each $u \in PBE(R)$, $|u - \psi u| \leqslant p_u$, for some potential p_u on R. Equivalently, $u|\Delta = \psi u|\Delta$, for every $u \in PBE(R)$. As a consequence of the homeomorphism between Δ^P and $\delta_P - \{s_P\}$, we will see that δ_P suffers from a similar limitation in being able to characterize $PBE(R)$.

2. Let R be a hyperbolic Riemann surface, and $M(R)$ the Royden algebra of bounded, Tonelli, Dirichlet-finite functions on R. Denote by $M_\Delta(R)$ the BD-closure of functions in $M(R)$ with compact supports. The Royden compactification R^* is a compact Hausdorff space containing R as an open dense subset such that every $f \in M(R)$ has a continuous extension to R^*.

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Moreover, $M(R)$ separates points of R^*.

The Royden boundary is $\Gamma = R^* \setminus R$, and the Royden harmonic boundary Δ can be characterized by

$$\Delta = \{x \in R^*: f(x) = 0 \text{ for all } f \in M_\Delta(R)\},$$

and hence is compact. Refer to the monograph [7] for a comprehensive treatment of the Royden algebra and Royden compactification.

For a nonnegative locally Hölder continuous differential $P = P(z)\,dx\,dy$, $z = x + iy$, we consider the P-algebra $M_P(R)$ of bounded, Tonelli, energy-finite functions f on R, where the energy $E_R(f)$ is given by

$$E_R(f) = \int_R |\text{grad}\, f|^2 + \int_R Pf^2.$$

Let $M_{P\Delta}(R)$ be the family of functions which are BE-closures of functions in $M_P(R)$ with compact supports. The P-compactification R_P^* is a compact Hausdorff space containing R as an open dense subset such that every $f \in M_P(R)$ has a continuous extension to R_P^*. Furthermore, $M_P(R)$ separates points of R_P^*.

The P-boundary is $\gamma_P = R_P^* \setminus R$, and the P-harmonic boundary δ_P can be characterized by

$$\delta_P = \{x \in R_P^*: f(x) = 0 \text{ for all } f \in M_{P\Delta}(R)\}.$$

The P-singular point $s_P \in \delta_P$ enjoys the property that $f(s_P) = 0$ for each $f \in M_P(R)$. It exists, and is unique, if and only if $\int_R P = \infty$. Moreover, under the condition $\int_R P = \infty$, $M_P(R) \subsetneq M(R)$ and $R_P^* \neq R^*$.

Of prime importance in the sequel will be the following:

URYSOHN PROPERTY FOR R^* (resp. R_P^*): *For disjoint compact subsets K_0, K_1 of R^* (resp. R_P^* such that K_0 contains the P-singular point s_P), there exists a function $f \in M(R)$ (resp. $M_P(R)$) such that $0 \leqslant f \leqslant 1$ on R^* (resp. R_P^*), and $f|K_i = i$, $i = 0, 1$.*

Denoting by $PBE(R)$ the space of bounded, energy-finite solutions of $\Delta u = Pu$ on R, the orthogonal decomposition obtains for every $f \in M_P(R)$:

$$f = u_f + g,$$

where $u_f \in PBE(R)$ is the *P-harmonic projection of f*, and $g \in M_{P\Delta}(R)$. The P-algebra and P-compactification have been extensively investigated in the works [4], [5], [6], and from a somewhat different point of view in [3].

Next, we can define a continuous open mapping π_P: $R^* \to R_P^* \subset \Pi_{f \in M_P(R)}[-\|f\|_\infty, \|f\|_\infty]$ of R^* onto R_P^* by $(\pi_P(x))_f = f(x)$, for $x \in R^*$, $f \in M_P(R)$. Then π_P is the identity mapping on R, and $f(\pi_P(x)) = f(x)$ for all $x \in R^*, f \in M_P(R)$.

In what follows, we prove that $\delta_P - \{s_P\}$ is the image under π_P of the set of nondensity points $\Delta^P \subset \Delta$, defined by

$$\Delta^P = \left\{x \in \Delta: \int_{U^* \cap R} P < \infty \text{ for some neighborhood } U^* \text{ of } x \text{ in } R^*\right\}.$$

3. Points of Γ which are mapped to s_P by π_P have the following characterization (cf. [8]).

THEOREM 1. *$\pi_P(x) = s_P$ if and only if $\int_{U^* \cap R} P = \infty$ for every neighborhood U^* of x in R^*.*

PROOF. Suppose $\pi_P(x) = s_P$ for some $x \in \Gamma$, and let U^* be a neighborhood of x in R^*. By the Urysohn Property for R^*, there exists a function $g \in M(R)$ such that $0 \leqslant g \leqslant 1$ on R^*, $g(x) = 1$, and $\operatorname{supp} g \subset U^*$. Then $g \in M_P(R)$ since every $f \in M_P(R)$ satisfies $f(x) = f(s_P) = 0$. As a consequence, $E_R(g) = \infty$, and therefore $\int_R Pg^2 = \infty$ since $g \in M(R)$. Hence

$$\int_{U^* \cap R} P \geqslant \int_{U^* \cap R} Pg^2 \geqslant \int_R Pg^2 = \infty,$$

and the necessity is proved.

On the other hand, suppose $\pi_P(x) = y \neq s_P$. We choose a function $f \in M_P(R)$ such that $f(y) > \varepsilon$ for some $\varepsilon > 0$. Then $N^* = \{z \in R_P^*: f(z) > \varepsilon\}$ is an open neighborhood of y in R_P^* with

$$\varepsilon^2 \int_{N^* \cap R} P \leqslant \int_{N^* \cap R} Pf^2 \leqslant E_R(f) < \infty.$$

Consider a neighborhood U^* of x in R^* such that $\pi_P(U^*) \subset N^*$. Since $U^* \cap R \subset \pi_P(U^*) \cap R$, we conclude

$$\int_{U^* \cap R} P \leqslant \int_{\pi_P(U^*) \cap R} P \leqslant \int_{N^* \cap R} P < \infty,$$

establishing the theorem.

COROLLARY. *If $\Delta^P \subsetneqq \Delta$, then $\pi_P(\Delta \setminus \Delta^P) = \{s_P\}$.*

The relationship between Δ and δ_P was initially studied in [9]. The following result establishes the relationship between Δ^P and $\delta_P - \{s_P\}$.

THEOREM 2. *$\pi_P: \Delta^P \to \delta_P - \{s_P\}$ is a homeomorphism.*

PROOF. We first show that $\pi_P(\Delta^P) \subset \delta_P - \{s_P\}$. Let $x \in \Delta$, and consider a function $f \in M_{P\Delta}(R)$. Since $M_{P\Delta}(R) \subset M_\Delta(R)$, $f(x) = 0$ by the characterization of Δ, and since $f(x) = f(\pi_P(x))$, we infer $\pi_P(x) \in \delta_P$. Therefore $\pi_P(\Delta) \subset \delta_P$. For any $x \in \Delta^P$, Theorem 1 implies $\pi_P(x) \neq s_P$, so that $\pi_P(\Delta^P) \subset \delta_P - \{s_P\}$.

To prove π_P is injective, take $x, y \in \Delta^P$, $x \neq y$. Then $\{\pi_P(x)\} \cup \{\pi_P(y)\}$ is a compact subset of $\delta_P - \{s_P\}$. By the Urysohn Property for R_P^*, there exists a function $g \in M_P(R)$ such that $0 \leqslant g \leqslant 1$ on R_P^*, and $g|\{\pi_P(x)\} \cup \{\pi_P(y)\} = 1$. Thus $g|\{x\} \cup \{y\} = 1$. Choose $h \in M(R)$ such that $h(x) \neq h(y)$, and $0 \leqslant h \leqslant 1$. The function $f = gh$ satisfies $f(x) \neq f(y)$ and $0 \leqslant f \leqslant g$ on R. Hence $f \in M_P(R)$, and $f(\pi_P(x)) \neq f(\pi_P(y))$ implies $\pi_P(x) \neq \pi_P(y)$.

To show π_P is surjective, suppose $x \in \delta_P - \{s_p\} \setminus \pi_P(\Delta)$. By the Urysohn

Property for R_P^*, there exists a function $f \in M_P(R)$ such that $0 \leqslant f \leqslant 1$ on R_P^*, $f(x) = 1$, and $f|\pi_P(\Delta) \cup \{s_p\} = 0$. Then the P-harmonic projection $u_f \in PBE(R)$ satisfies $u_f|\delta_P = f|\delta_P$. Thus $u_f(x) = 1$, but $u_f|\Delta = u|\pi_P(\Delta) = 0$, implying $u_f \equiv 0$ by the PBD-Maximum Principle (cf. [2]). This contradicts $u_f(x) = 1$, and hence $\delta_P - \{s_P\} \subset \pi_P(\Delta)$. For any $z \in \Delta \setminus \Delta^P$, Theorem 1 implies $\pi_P(z) = s_P$, so that $\delta_P - \{s_P\} \subset \pi_P(\Delta^P)$, and the theorem is proved.

From the preceding it is evident that:

Corollary 1. $\pi_P(\Delta) = \delta_P - \{s_P\}$ *if and only if* $\Delta = \Delta^P$.

Corollary 3.3 of [2] can now be stated in the following manner:

Corollary 2. *$HBD(R)$ is canonically isomorphic to $PBE(R)$ if and only if* $\pi_P(\Delta) = \delta_P - \{s_P\}$.

4. A necessary condition for $PBE(R)$ to be canonically isomorphic to $QBE(R)$ is that $\Delta^P = \Delta^Q$ (cf. [2]). In this case, $\pi_Q \circ \pi_P^{-1}$ is a homeomorphism between $\delta_P - \{s_P\}$ and $\delta_Q - \{s_Q\}$. Since δ_P and δ_Q are compact, we can extend $\pi_Q \circ \pi_P^{-1}$ to a homeomorphism between δ_P and δ_Q, yielding:

Theorem 3. *If $PBE(R)$ is canonically isomorphic to $QBE(R)$, then δ_P is homeomorphic to δ_Q.*

To disprove the converse, consider the Riemann surface T^∞ constructed in [2] which has the property that for certain densities P and Q on R, $\Delta^P = \Delta^Q$, however, $PBE(T^\infty)$ is not canonically isomorphic to $QBE(T^\infty)$. The densities P and Q satisfy $\int_R P = \int_R Q = \infty$, so that $R_P^* \neq R^*$, $R_Q^* \neq R^*$. Thus we see that δ_P does not characterize the space $PBE(R)$ in the following sense:

Theorem 4. *There exists a Riemann surface R and densities P and Q on R such that δ_P is homeomorphic to δ_Q, but $PBE(R)$ is not canonically isomorphic to $QBE(R)$.*

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