PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 66, Number 1, September 1977

A NECESSARY AND SUFFICIENT CONDITION THAT A FUNCTION ON THE MAXIMAL IDEAL SPACE OF A BANACH ALGEBRA BE A MULTIPLIER

JAMES A. WOOD

ABSTRACT. Consider a regular commutative, semisimple Banach algebra with a bounded approximate identity whose Gelfand transforms have compact support. A necessary and sufficient condition is given for a complex valued function defined on the maximal ideal space to determine a multiplier of the algebra. This theorem is similar to one proved by F. T. Birtel, but omits Birtel's assumption that the algebra be topologically embeddable in its second dual.

1. **Introduction.** Let A be a regular commutative semisimple Banach algebra and denote by $\Delta(A)$ the maximal ideal space of A endowed with the Gelfand topology. If $x \in A$, $\hat{x}$ will denote the Gelfand transform of x and $\hat{A}$ the algebra of all transforms. We assume further that A has a bounded approximate identity $\{e_n\}$ and that each $\hat{e}_n$ has compact support. By a multiplier of A we mean a bounded linear operator $F: A \to A$ such that $F(xy) = xF(y)$ for all $x, y \in A$.

If $\hat{F}$ is a complex valued function on $\Delta(A)$, we denote by A_F the set $\{x \mid \hat{F}\hat{x} = \hat{y} \text{ for some } y \in A\}$. The set A_F is always nonempty since it contains at least $x = 0$. Moreover, A_F is a subspace of A, and we can define a linear function $F: A_F \to A$ by $F(x) = y$ if and only if $\hat{F}\hat{x} = \hat{y}$. If $A_F = A$, then an application of the closed graph theorem shows that F is a bounded linear operator, and it is obvious that F is a multiplier of A. Conversely, it is also well known that if F is a multiplier of A, then there is a unique function $\hat{F}$ on $\Delta(A)$ such that $F(x)\hat{} = \hat{F}\hat{x}$ for all x. This correspondence is actually an isomorphism.

It is the purpose of this note to prove a necessary and sufficient condition that a function $\hat{F}$ on $\Delta(A)$ determine a multiplier F of A. This theorem is related to a theorem proved by F. T. Birtel in 1962 in [2, p. 819]. The main difference between our result and Birtel's theorem is that we are able to replace Birtel's assumption that A be topologically embeddable in A'' with a different assumption which does not involve A''. Here $A'' = (A')^*$, where A' is the closed linear span of $\Delta(A)$ in A^*, the conjugate space of A.

In what follows we need to make use of the fact that A^{**} is a Banach

Received by the editors July 13, 1976 and, in revised form, December 27, 1976 and January 27, 1977.

AMS (MOS) subject classifications (1970). Primary 46J99.

algebra under a multiplication introduced by R. Arens [1], or [2, p. 816]. For the sake of completeness, we sketch the definition of this multiplication. If $p \in A^*$, $x \in A$, define $px(y) = p(xy)$. It is easy to check that $px \in A^*$ and $\|px\| \leqslant \|p\|\,\|x\|$. If $\Phi \in A^{**}$, $p \in A^*$, define Φp by $\Phi p(x) = \Phi(px)$. Again it is easy to check that $\Phi p \in A^*$ and $\|\Phi p\| \leqslant \|\Phi\|\,\|p\|$. Therefore, if $\Phi, \Psi \in A^{**}$, define $\Phi\Psi(p) = \Phi(\Psi p)$, $p \in A^*$. Finally, $\Phi\Psi$ is linear on A^* and $\|\Phi\Psi\| \leqslant \|\Phi\|\,\|\Psi\|$, so $\Phi\Psi \in A^{**}$.

2. **Proof of the Main Theorem**. Before proving the main theorem we first need a lemma.

LEMMA. *Suppose $\hat{F}$ is a complex valued function defined on $\Delta(A)$ and that for some $x \in A$, $\hat{F}\hat{x} \in \hat{A}$. Let y be that element in A such that $\hat{y} = \hat{F}\hat{x}$ and write $y = F(x)$. Assume $|p(y)|/\|px\| = |p(F(x))|/\|px\| \leqslant M$ for all $p \in A^*$. Then there exists a $\Phi \in A^{**}$ such that $y^{**} = (F(x))^{**} = \Phi x^{**}$, where the product Φx^{**} denotes the Arens product in A^{**}.*

PROOF. Let $x^{**}A^* = \{x^{**}p \mid p \in A^*\}$. It is routine to check that $x^{**}p + x^{**}q = x^{**}(p + q)$ and that $\alpha(x^{**}p) = x^{**}\alpha p$ so that $x^{**}A^*$ is a subspace of A^*. We define Φ_0 on $x^{**}A^*$ by $\Phi_0(x^{**}p) = p(F(x))$. Now

$$\begin{aligned}\Phi_0(x^{**}p + x^{**}q) &= \Phi_0(x^{**}(p + q)) = (p + q)(F(x)) \\ &= p(F(x)) + q(F(x)) = \Phi_0(x^{**}p) + \Phi_0(x^{**}q).\end{aligned}$$

Also

$$\Phi_0(\alpha x^{**}p) = \Phi_0(x^{**}\alpha p) = \alpha p(F(x)) = \alpha\Phi_0(x^{**}p).$$

Thus Φ_0 is linear on $x^{**}A^*$. Moreover,

$$\frac{|\Phi_0(x^{**}p)|}{\|x^{**}p\|} = \frac{|p(F(x))|}{\|px\|} \leqslant M,$$

so $\|\Phi_0\| \leqslant M$ and Φ_0 is bounded. By the Hahn-Banach theorem Φ_0 can be extended to a functional Φ on all of A^* having the same norm as Φ_0. Now

$$(F(x))^{**}(p) = p(F(x)) = \Phi_0(x^{**}p) = \Phi(x^{**}p) = \Phi x^{**}(p)$$

for all $p \in A^*$, so that $(F(x))^{**} = \Phi x^{**}$.

We can now prove our main result.

THEOREM. *Let $\hat{F}$ be a complex valued function on $\Delta(A)$. In order that $\hat{F}$ determine a multiplier of A, it is necessary and sufficient that $\hat{F}$ belongs locally to $\Delta(A)$ at each point of $\Delta(A)$ and that $|p(F(x))|/\|px\| \leqslant M$, for all $p \in A^*$ and for all $x \in A_F$.*

PROOF. We show first that the condition is sufficient. In [2, p. 818], Birtel showed that if a function f on $\Delta(A)$ belongs locally to $\hat{A}$ at each $p \in \Delta(A) \cup \{\infty\}$, then $f \in \hat{A}$. From this result it follows that $\hat{F}\hat{x}\hat{e}_n \in \hat{A}$ for all $x \in A$ and all n, i.e. $xe_n \in A_F$. By the lemma, we know that $(F(xe_n))^{**} = \Phi_{nm}(xe_n)^{**}$, where $\|\Phi_{nm}\| \leqslant M$ and M is independent of x and Φ in general depends upon F and xe_n. Thus

$$\begin{aligned}\|F(xe_n) - F(xe_m)\| &= \|(F(xe_n) - F(xe_m))^{**}\| \\ &= \|\Phi_{nm}(xe_n - xe_m)^{**}\| \\ &\leqslant \|\Phi_{nm}\| \, \|(xe_n - xe_m)^{**}\| \\ &\leqslant M\|xe_n - xe_m\|.\end{aligned}$$

Therefore, $\{F(xe_n)\}$ is a Cauchy net in A. Let $y = \lim F(xe_n)$ and define $F(x) = y$. Clearly, $\hat{F}\hat{x} = \hat{y}$ so that $\hat{F}$ determines the multiplier F.

To prove the necessity suppose $\hat{F}$ determines the multiplier F. It is an easy consequence of the regularity of A that $\hat{F}$ belongs locally to $\hat{A}$ at points of $\Delta(A)$. To obtain the desired estimate we observe first that

$$\begin{aligned}|p(F(xe_n))| = |p(xF(e_n))| &= |px(F(e_n))| \\ &\leqslant \|px\| \, \|F\| \, \|e_n\| \leqslant \|px\| M.\end{aligned}$$

Therefore, $|p(F(xe_n))|/\|px\| \leqslant M$. But

$$p(F(xe_n)) = p(e_n F(x)) \to p(F(x))$$

so that $|p(F(x))|/\|px\| \leqslant M$ for all $x \in A$ and all $p \in A^*$.

References

1. R. Arens, *Operations induced on function class*, Monatsh. Math. **55** (1951), 1–19.

2. F. T. Birtel, *On a commutative extension of a Banach algebra*, Proc. Amer. Math. Soc. **13** (1962), 815–822.

Department of Mathematics, Virginia Commonwealth University, Richmond, Virginia 23284