UNITARY PARTS OF GENERALIZED TOEPLITZ OPERATORS

JEFFREY R. BUTZ

ABSTRACT. For a generalized Toeplitz operator acting on a Hilbert space with unilateral shift, necessary and sufficient conditions are obtained for the existence of a nontrivial unitary part, and an explicit description of this unitary part is given.

Let T be a contractive linear operator on a separable complex Hilbert space X . Such an operator is said to be *completely nonunitary* if it has no nontrivial reducing subspace N such that the restriction $T|_N$ of T to N is unitary. It is known [2, Theorem I.3.2] that for any contraction T on X we can find a unique orthogonal decomposition $X = M \oplus M_1$ such that M and M_1 reduce T , $T|_M$ is unitary and $T|_{M_1}$ is completely nonunitary. It is not excluded that M or M_1 is possibly the subspace $\{0\}$. Furthermore, M is given by $M = \{x \in X: \|T^n x\| = \|x\| = \|T^{*n} x\|, n = 1, 2, \dots\}$ and is called the *unitary subspace* of T . $T|_M$ is called the *unitary part* of T .

In this paper, we describe explicitly the above decomposition for the class of generalized Toeplitz operators, which are obtained as follows.

We say that an isometry $S: X \rightarrow X$ is a *unilateral shift* if there exists a subspace C in X for which $(S^j C) \perp (S^k C)$ for nonnegative integers $j \neq k$, and

$$X = \bigoplus_{n=0}^{\infty} S^n C.$$

It is not difficult to see that C is uniquely determined by S , viz., $C = (SX)^\perp$.

In what follows, we shall find it convenient to consider the minimal unitary extension U of S . Then U is a bilateral shift acting on a Hilbert space Y containing X , and Y decomposes into the orthogonal direct sum

$$Y = \bigoplus_{n=-\infty}^{\infty} U^n C \quad \text{and} \quad U|_X = S.$$

That such a space Y exists is clear, since it can be constructed as a direct sum of a countable number of copies of C indexed on the integers. Equivalently, U and Y are obtained as the minimal unitary dilation of S in the structure theory of Sz.-Nagy and Foiaş [2, Chapter I].

We now fix a unilateral shift S on the Hilbert space X and make the following definitions.

DEFINITIONS. 1. A bounded operator $T: X \rightarrow X$ is *Toeplitz* if $S^* T S = T$.

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2. A bounded operator $L: Y \rightarrow Y$ is *Laurent* if $LU = UL$.

These operators have been previously studied by Rosenblum and Page, cf. [4], and they are generalizations of the classical Toeplitz and Laurent operators, where X is the Hardy space H^2 consisting of those L^2 functions defined on the unit circle whose Fourier coefficients vanish on the negative integers and S is equal to multiplication by z . In particular, we have the following fact which, in this generalized setting, has been observed by Page [3].

PROPOSITION (PAGE [3]). *A bounded operator T on X is Toeplitz if and only if there exists a bounded Laurent operator L on Y such that $T = P_+ L|_X$, where $P_+: Y \rightarrow X$ is the orthogonal projection. In this case, $\|L\| = \|T\|$.*

This generalizes the fact that, in the classical case, Toeplitz operators are of the form $T_\varphi: x(e^{i\theta}) \mapsto P_+ \varphi(e^{i\theta}) x(e^{i\theta})$, where $\varphi \in L^\infty$ and $P_+: L^2 \rightarrow H^2$ is the orthogonal projection. In such a case, we have $\|T_\varphi\| = \|\varphi\|_\infty$ and $T_\varphi^* = T_{\bar{\varphi}}$. For these classical Toeplitz operators, Goor [1] has proved the following.

THEOREM (GOOR [1]). *If $\varphi \in L^\infty$, $\|\varphi\|_\infty \leq 1$, and φ is (almost everywhere) nonconstant, then T_φ is completely nonunitary.*

We now obtain a theorem which generalizes to abstract Toeplitz operators this characterization of unitary parts. Unlike classical Toeplitz operators, however, nontrivial abstract Toeplitz operators can have nontrivial unitary parts. A simple example of this is given by $X = H^2 \oplus H^2$ with $T: X \rightarrow X$ defined by $T = T_0 \oplus T_1$, where $T_0: H^2 \rightarrow H^2$ is multiplication by a constant of modulus 1, and $T_1: H^2 \rightarrow H^2$ is a completely nonunitary Toeplitz operator. Then T has the nontrivial unitary subspace $M = H^2 \oplus \{0\}$, and the restriction of T to M is multiplication by a constant of modulus 1 (which is a unitary operator on the 1-dimensional generating subspace for M). Our theorem shows that all abstract Toeplitz contractions decompose in a similar way.

THEOREM. *Let $T: X \rightarrow X$ be a Toeplitz contraction on the Hilbert space $X = \bigoplus_{n=0}^\infty S^n C$. Then there exists a decomposition $C = C_0 \oplus C_1$ such that $M = \bigoplus_{n=0}^\infty S^n C_0$, $M^\perp = \bigoplus_{n=0}^\infty S^n C_1$, where M is the unitary subspace of T , and $x = \sum_{n=0}^\infty S^n c_n \in M \Leftrightarrow c_n \in C_0$ for $n = 0, 1, 2, \dots$. Furthermore, if $M \neq \{0\}$ (i.e., $C_0 \neq \{0\}$), then there exists a unitary operator $R_0: C_0 \rightarrow C_0$ such that for $x = \sum_{n=0}^\infty S^n c_n \in M$ we have $Tx = \sum_{n=0}^\infty S^n R_0 c_n$.*

PROOF. By the above proposition, there exists an operator $L: Y \rightarrow Y$ such that $P_+ Lx = Tx$ for $x \in X$, $\|L\| = \|T\| \leq 1$, and $UL = LU$. Suppose that there exists a nonzero x in the unitary subspace M of T , which, as indicated above, is given by $M = \{x \in X: \|T^n x\| = \|x\| = \|T^{*n} x\|, n = 1, 2, \dots\}$. Then, in particular,

$$\|x\| = \|Tx\| = \|P_+ Lx\| \leq \|Lx\| \leq \|x\|,$$

so that $Lx = Tx \in X$. Similarly, $L^*x = T^*x \in X$. Continuing for $n = 2, 3, \dots$, we obtain

$$(1) \quad M = \{x \in X: L^n x, L^{*n} x \in X \text{ and } \|L^n x\| = \|x\| = \|L^{*n} x\|, \\ n = 1, 2, \dots\}.$$

Also, for $x \in M$ we have $L^n Sx = SL^n x \in X$ and $\|L^n Sx\| = \|SL^n x\| = \|x\| = \|Sx\|$, and similarly for L^{*n} . Therefore, the subspace M is invariant under the unilateral shift. Then the restriction of S to M is a shift since

$$\bigcap_{k=0}^{\infty} S^k M \subseteq \bigcap_{k=0}^{\infty} S^k X = \{0\},$$

and thus we can write $M = \bigoplus_{n=0}^{\infty} S^n C_0$, where $C_0 = (SM)^{\perp} \cap M$. We claim that the subspace C_0 is reducing for L (and hence T). Let $c \in C_0$, then by (1) $c \in M \Rightarrow Lc \in M$, and for $x \in M$ we have $\langle Lc, Sx \rangle = \langle c, L^* Sx \rangle = \langle c, SL^* x \rangle = 0$, since $SL^* x \in SM$ and $c \in (SM)^{\perp}$. Therefore, $Lc \in M$ and similarly $L^* c \in M$, so that C_0 reduces L as claimed.

Writing now $x = \sum_{n=0}^{\infty} S^n c_n \in M$, $c_n \in C_0$, we obtain

$$Tx = Lx = L\left(\sum_{n=0}^{\infty} S^n c_n\right) = \sum_{n=0}^{\infty} S^n Lc_n = \sum_{n=0}^{\infty} S^n R_0 c_n,$$

where $R_0 = L|_{C_0}$. Also, for $S^* x = \sum_{n=0}^{\infty} S^n c_{n+1}$ we have

$$LS^* x = L\left(\sum_{n=0}^{\infty} S^n c_{n+1}\right) = \sum_{n=0}^{\infty} S^n Lc_{n+1} = \sum_{n=0}^{\infty} S^n R_0 c_{n+1} \in X,$$

so that $\|TS^* x\| = \|LS^* x\| = \|S^* x\|$, and similarly for T^* and powers $n = 2, 3, \dots$. Hence, M is in fact a reducing subspace for the shift.

We now claim that $C_0 = C \cap M$. It is clear that $C \cap M = (SX)^{\perp} \cap M \subseteq (SM)^{\perp} \cap M = C_0$, so we must show that if $x \in (SM)^{\perp} \cap M$, then $x \in (SX)^{\perp} \cap M$. For $y \in X$ write $y = y_1 + y_2$ with $y_1 \in M$, $y_2 \in M^{\perp}$, then

$$\langle x, Sy \rangle = \langle x, Sy_1 \rangle + \langle x, Sy_2 \rangle = 0 + \langle S^* x, y_2 \rangle = 0,$$

so that $x \in (SX)^{\perp} \cap M$ as required.

It remains to show that for $C_1 = C \ominus C_0$ we have $M^{\perp} = \bigoplus_{n=0}^{\infty} S^n C_1$. If $\sum_{n=0}^{\infty} S^n c'_n \in \bigoplus_{n=0}^{\infty} S^n C_1$ and $\sum_{m=0}^{\infty} S^m c_m \in M$, $c'_n \in C_1$, $c_m \in C_0$, then

$$\left\langle \sum_{n=0}^{\infty} S^n c'_n, \sum_{m=0}^{\infty} S^m c_m \right\rangle = \sum_{m=0}^{\infty} \langle S^n c'_n, S^n c_n \rangle = \sum_{n=0}^{\infty} \langle c'_n, c_n \rangle = 0,$$

so that $\bigoplus_{n=0}^{\infty} S^n C_1 \subseteq M^{\perp}$. Conversely, suppose that we had $x = \sum_{n=0}^{\infty} S^n c_n \in M^{\perp}$ with $c_k \notin C_1$ for some integer k . Then write $c_k = c'_k + c''_k$ with $c'_k \in C_0$, $c''_k \in C_1$, $c'_k \neq 0$. Since $S^k c'_k \in M$, we would then get

$$\langle x, S^k c'_k \rangle = \left\langle \sum_{n=0}^{\infty} S^n c_n, S^k c'_k \right\rangle = \langle S^k c_k, S^k c'_k \rangle = \langle c'_k, c'_k \rangle \neq 0,$$

which contradicts that $x \in M^\perp$. Therefore, $M^\perp = \bigoplus_{n=0}^{\infty} S^n C_1$, and the proof is complete.

REMARKS. 1. As a converse to the above theorem, note that any orthogonal decomposition $C = C_0 \oplus C_1$ together with a unitary operator $R_0: C_0 \rightarrow C_0$ determines a Toeplitz operator T whose unitary subspace is given by $M = \bigoplus_{n=0}^{\infty} S^n C_0$. For example, define T by

$$T \left(\sum_{n=0}^{\infty} S^n c_n \right) = T \left(\sum_{n=0}^{\infty} S^n (c'_n + c''_n) \right) = \sum_{n=0}^{\infty} S^n R_0 c'_n,$$

where $c_n = c'_n + c''_n \in C$, $c'_n \in C_0$, $c''_n \in C_1$.

2. The special case of $\dim C = 1$, as described by Goor, follows as an easy consequence of the theorem. Either $\dim C_0 = 0$, so that $M = \{0\}$ and T is completely nonunitary, or $\dim C_0 = 1$ and $X = \bigoplus_{n=0}^{\infty} S^n C_0 = M$. In the latter case, for $x \in X$, $x = \sum_{n=0}^{\infty} S^n c_n$ for some $c_n \in C_0$, we have by the theorem that $Tx = \sum_{n=0}^{\infty} S^n R_0 c_n$, where R_0 is a unitary operator on C_0 . But then R_0 must be multiplication by a scalar α of modulus 1, which implies that T is itself multiplication by the scalar α , as asserted in Goor's result.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OKLAHOMA, NORMAN, OKLAHOMA 73019