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OPEN AND UNIFORMLY OPEN RELATIONS[1]

P. MAH AND S. A. NAIMPALLY

ABSTRACT. It is shown that if (X, δ) is an Efremovič proximity space, Y is a topological space, $R \subset X \times Y$ is an injective relation, then R is open if and only if R is weakly open, nearly open and $R[X]$ is open in Y. An analogous result is proved when X is a uniform space and Y a Morita uniform space: (i) if R is uniformly open, then R is weakly open and uniformly nearly open; (ii) if R is weakly open, and uniformly nearly open, then R is uniformly open on X to $R[X]$. These results include, as special cases, results of Kelley, Pettis and Weston.

Our investigation began in an attempt to find a common refinement of a uniformly open mapping theorem of Kelley [1, Lemma 6.36], and an open mapping theorem of Weston [7, Theorem 8]. This problem was posed by Pettis [5] at the Topology Conference at Charlotte, N. C. In this paper, we prove two very general results concerning (uniform) openness of a relation (Theorems 3 and 7). From these two results, we derive generalizations of several known results. In particular, we derive generalizations of the results of Kelley and Weston, the former to Morita range space, and the latter to point-compact upper semicontinuous relations. This latter generalization of Weston's result allows us to both generalize (to relations) and dualize, in a certain sense, some recent results of Pettis. Finally, we give an example which shows that Weston's result cannot be generalized to a cocompact domain. This answers a question raised in [5].

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Let X and Y be two topological spaces. We shall consider two types of proximities, Efremovič (or EF-proximity) and Lodato (or LO-proximity). Frequently one of the spaces considered is a metric space; in that case, if the metric is d, we shall suppose that the proximity considered is the EF-proximity δ induced by d viz. $A \ \delta \ B$ iff $d(A, B) = \inf\{d(a, b): a \in A, b \in B\} = 0$. (Read $A \ \delta \ B$ as "A is near B"; its negation "A is far from B" is written as $A \not\delta B$.) Notations and results of [3] will be freely used in this paper.

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To obtain a generalization of Kelley's result, we shall assume the range space Y of a relation to be an R_0-space (or weakly regular) i.e. $x \in \{y\}^-$ implies $y \in \{x\}^-$ for all $x, y \in Y$, where $^-$ denotes the closure operator. For such a space, Morita [2] has shown that there is a family Ω (called a *Morita uniformity*) of open covers of Y such that $\{\alpha^*(y): \alpha \in \Omega\}$ is a neighbourhood base at y for each $y \in Y$, where $\alpha^*(y) = \bigcup\{G \in \alpha: y \in G\}$.

A relation $R \subset X \times Y$ is *injective* if $A \cap B = \varnothing$ in X implies $R[A] \cap R[B] = \varnothing$ in Y. We note that if R is a function from X to Y, the inverse relation R^{-1} is always injective, and conversely, if R is a relation on X to Y such that R^{-1} is injective, then R is a function. A relation is called *bijective* if it is injective and onto. A relation R is called (i) *open* (resp. *closed*) iff for each open (resp. closed) set A in X, $R[A]$ is open (resp. closed) in Y, (ii) *nearly open* iff for each open set A in X, $R[A] \subset R[A]^{-0}$, (iii) *upper semicontinuous* (or USC) iff for each $x \in X$ and each open set V in Y such that $R(x) \subset V$, there is a neighbourhood U of x such that $R[U] \subset V$, (iv) *nearly continuous* (when R is a function) iff for each open set B in Y, $R^{-1}[B] \subset R^{-1}[B]^{-0}$, where ^{0} denotes the interior operator. If R is a closed subset of $X \times Y$, then we say that R has a *closed graph*.

If $(X, \mathfrak{U})$ is a uniform space, and (Y, Ω) a Morita uniform space, then $R \subset X \times Y$ is called (i) *uniformly open* iff for each $U \in \mathfrak{U}$ there is an $\alpha \in \Omega$ such that for all $(x, y) \in R$, $\alpha^*(y) \subset R[U[x]]$, (ii) *uniformly nearly open* iff for each $U \in \mathfrak{U}$, there is an $\alpha \in \Omega$ such that for all $(x, y) \in R$, $\alpha^*(y) \subset R[U[x]]^{-0}$. If (X, δ) is a LO-proximity space, then $R \subset X \times Y$ is called *weakly open* iff for each $A \subset X, y \in R[A]^- \cap R[X]$ implies $R^{-1}(y)\, \delta\, A$ (see Poljakov [6]). (For our purposes, it is sufficient that the above holds for each open set A.)

It is obvious that a (uniformly) open relation is (resp. uniformly) nearly open. However, contrary to an assertion in [6], an open relation need not be weakly open as the following example shows.

1. Example. Let X be the subset of $\mathbf{R}^2$ defined by $\{(x, y): y > 5\} \cup \{(x, y): x > 0 \text{ and } y > 1\}$, with the subspace proximity δ induced by the usual Euclidean metric. Let $f: X \to \mathbf{R}$ be the projection $f(x, y) = x$. Since X is open in $\mathbf{R}^2$, f is an open map. If $A = \{(x, y): x > 0, 1 < y < 2\}$ then $0 \in f(A)^-$ but $f^{-1}(0) = \{(0, y): y > 5\} \not\delta A$. Thus f is not weakly open.

If we require the relation to be *injective*, then the assertion of [6] is true.

2. Lemma. *If (X, δ) is a LO-proximity space, Y a topological space, and $R \subset X \times Y$ is an open injective relation, then R is weakly open.*

Proof. If the result is not true, then there exist $A \subset X$, $y \in R[A]^- \cap R[X]$ such that $x \notin A^-$, where $x = R^{-1}(y)$ (noting that R^{-1} is a function since R is injective). Thus, there is an open set U in X such that $x \in U$ and $U \cap A = \varnothing$. Since R is injective and open, $R[U]$ is an open set containing y and disjoint from $R[A]$, a contradiction.

In the opposite direction, we have the following fundamental result of this paper.

3. THEOREM. *If (X, δ) is an EF-proximity space, Y is a topological space, $R \subset X \times Y$ is a weakly open and nearly open relation, then R is open on X to $R[X]$.*

PROOF. We must show that $R[V]$ is open in $R[X]$ for each open set V in X. Let $y \in R[V]$. Then there is an $x \in V$ with $(x, y) \in R$. Since $x \not\delta (X \sim V)$, there is a neighbourhood U of x with $U \not\delta (X \sim V)$. Since R is nearly open, $R[U]^-$ is a neighbourhood of y, and so the proof is complete once we show that $R[U]^- \cap R[X] \subset R[V]$. Suppose, then, that $z \in R[U]^- \cap R[X]$. Since R is weakly open, $R^{-1}(z)\ \delta\ U$, and since $U \not\delta (X \sim V)$, we have $R^{-1}(z) \cap V \neq \emptyset$. Thus $z \in R[V]$.

We note that if $R[X]$ is closed, then it is also open, since $R[X] \subset R[X]^{-0} = R[X]^0$. Thus, if $R[X]$ is either open or closed in Y, then under the hypothesis of Theorem 3, R is open.

4. COROLLARY. *If (X, δ) is an EF-proximity space, Y is a topological space, $R \subset X \times Y$ is an injective relation, then R is open iff R is weakly open, nearly open and $R[X]$ is open.*

We now derive a generalized version of a well-known result of Weston [7, Theorem 8]. A relation $R \subset X \times Y$ is called *separating* iff for each pair x_1, x_2 of distinct points of X, there exist neighbourhoods U_1, U_2 respectively such that $R[U_1]^- \cap R[U_2] = \emptyset = R[U_1] \cap R[U_2]^-$. Clearly if R is separating and the domain of R is X then X is Hausdorff. Also, if R is separating, then R is injective and $R(x)$ is closed in $R[X]$ for all x. Conversely, any one of the following conditions is sufficient for R to be separating:

(a) X is Hausdorff and R is open and injective;
(b) Y is Hausdorff, R is USC, injective and point-compact;
(c) Y is T_4, R is USC and injective.

When applied in case (b), the next theorem clearly yields Weston's result [7, Theorem 8].

5. THEOREM. *If (X, d) is a complete metric space, $R \subset X \times Y$ is separating, nearly open and $R[X]$ is open or closed in Y, then R is open.*

PROOF. In view of the remarks immediately following Theorem 3, it suffices to prove the theorem when $R[X]$ is open. Our proof is patterned after Weston's proof in [7]. By Corollary 4, it is sufficient to prove that R is weakly open. Suppose then that $A \subset X$, $y \in R[A]^- \cap R[X]$, $q_0 = R^{-1}(y)$ (a single point as R is injective). We must show that $q_0 \in A^-$. Take any $r_0 > 0$ and put $r_m = r_0/2^m$. Suppose that, for a given $n \geqslant 0$, q_m has been defined for all m, $0 \leqslant m \leqslant n$, and x_m for $0 \leqslant m < n$ such that the following conditions are satisfied:

(i) $x_m \in U(x_{m-1}, r_m)$ for $1 \leqslant m < n$;

(ii) $q_m \in U(q_{m-1}, r_{m-1})$ for $1 \leqslant m \leqslant n$;

(iii) $R[U(q_m, r_m)]^- \cap R[A_m] \neq \varnothing$ for $0 \leqslant m \leqslant n$, where $A_0 = A$ and $A_m = U(x_{m-1}, r_m)$ for $1 \leqslant m \leqslant n$.

(For $n = 0$, (i) and (ii) are vacuous, and (iii) is satisfied because, R being nearly open, $R[U(q_0, r_0)]^-$ is a neighbourhood of $R(q_0)$ and so of y, and hence intersects $R[A]$.)

Then by (iii), we can choose for $n \geqslant 0$, $x_n \in A_n$ with $R(x_n) \cap R[U(q_n, r_n)]^- \neq \varnothing$. This shows that (i) is satisfied for $m = n > 0$. Again $R[U(x_n, r_{n+1})]^-$ is a neighbourhood of $R(x_n)$ and hence intersects $R[U(q_n, r_n)]$. So we can choose $q_{n+1} \in U(q_n, r_n)$ with $R(q_{n+1}) \cap R[U(x_n, r_{n+1})]^- \neq \varnothing$. Thus (ii) is satisfied for $m = n + 1$. A similar application shows that (iii) is also satisfied for $m = n + 1$. Hence the construction is possible inductively for all m. Clearly, (x_m) and (q_m) form Cauchy sequences with respective limits x, q say. We claim that $x = q$; for if not, neighbourhoods N_x, N_q would exist as in the definition of a separating relation, and for sufficiently large m, $U(q_m, r_m) \subset N_q$ and $U(x_{m-1}, r_m) \subset N_x$, violating condition (iii). By construction $d(q, q_0) \leqslant 2r_0$ and $d(x, x_0) \leqslant r_0$; so as $x_0 \in A$ and r_0 is arbitrary, we have $q_0 \in A^-$ as required.

If f: $X \to Y$ is a function, then f is closed iff f^{-1} is USC; thus taking $R = f^{-1}$ in the above result we get the following.

6. COROLLARY. *Let X be T_2 (resp. T_4), Y a complete metric space, and f: $X \to Y$ a nearly continuous, closed function with compact (resp. no condition) fibers. Then f is continuous.*

We now consider uniformly (nearly) open relations in order to generalize Kelley's result, which is done in 8.

7. THEOREM. *Let $(X, \mathcal{U})$ be a uniform space, and (Y, Ω) be an R_0 Morita uniform space.*

(i) *If R is uniformly open, then R is weakly open and uniformly nearly open.*

(ii) *If R is weakly open and uniformly nearly open, then R is uniformly open on X to $R[X]$.*

Thus it follows that R is uniformly open iff R is weakly open, uniformly nearly open and $R[X]$ is either open or closed.

PROOF. (i) Suppose R is uniformly open. We must show that R is weakly open; if this is false then there exist $A \subset X$, $y \in R[A]^- \cap R[X]$, $U \in \mathcal{U}$ such that $U[A] \cap B = \varnothing$ where $B = R^{-1}(y)$. Thus, $R[U[A]] \cap R[B] = R[U[A]] \cap \{y\} = \varnothing$ (for otherwise, $y \in R[U[A]]$ and so $B \cap U[A] \neq \varnothing$, a contradiction). Since R is uniformly open, there exists an $\alpha \in \Omega$ such that for all $(x, y) \in R$, $\alpha^*(y) \subset R[U[x]]$, and so $\alpha^*(R[A]) \subset R[U[A]]$, where $\alpha^*(Z) = \bigcup \{\alpha^*(z)\colon z \in Z\}$. Thus, $y \notin \alpha^*(R[A])$. Since Y is R_0, $\alpha^*(y) \cap R[A] = \varnothing$, contradicting $y \in R[A]^-$. Clearly R is uniformly nearly open if it is uniformly open.

(ii) Assume R is uniformly nearly open and weakly open. The former property assures us that for every $U \in \mathfrak{U}$, there is an $\alpha \in \Omega$ such that for all $(x, y) \in R$, $\alpha^*(y) \subset R[V[x]]^-$, where $V \in \mathfrak{U}$ with $V^2 \subset U$. To prove R is uniformly open in $R[X]$, we show $R[V[x]]^- \cap R[X] \subset R[U[x]]$. Suppose then $z \in R[V[x]]^- \cap R[X]$. Since R is weakly open, $R^{-1}(z)\ \delta\ V[x]$. But $V[x] \not\delta (X \sim U[x])$, and so $R^{-1}(z) \cap U[x] \neq \varnothing$ i.e. $z \in R[U[x]]$.

8. Corollary. *Let (X, d) be a complete metric space, (Y, Ω) an R_0 Morita uniform space, and $R \subset X \times Y$ a relation with a closed graph. If R is uniformly nearly open, then R is uniformly open.*

Proof. This proof is patterned after Kelley [1, Lemma 6.36]. It is sufficient to show that for any subset $A \subset X$, if there is a $p \in Y$ such that $p \in R[A]^-$, then $R^{-1}(p)\ \delta\ A$. This will not only show that R is weakly open, but also that $R[X]$ is closed in Y. This latter implication follows by choosing $A = X$, and noting that, if $p \in R[X]^-$, then $R^{-1}(p)\ \delta\ X$ and so $R^{-1}(p) \neq \varnothing$. Since R is uniformly nearly open, it then follows that $R[X]$ is open; Theorem 7 then can be applied to conclude that R is uniformly open.

Let, then, $A \subset X, p \in R[A]^-$, and $\varepsilon > 0$. There is an $\alpha \in \Omega$ such that for all $(x, y) \in R$, $\alpha^*(y) \subset R[U(x, \varepsilon/2^2)]^-$. Since $p \in R[A]^-$, $\alpha^*(p) \cap R[A] \neq \varnothing$, and so there exists $(u, v) \in R$ such that $u \in A$ and $v \in \alpha^*(p)$. Then $p \in \alpha^*(v) \subset R[U(u, \varepsilon/2^2)]^-$. We now choose inductively a sequence of subsets A_n, $n \geqslant 0$, with the properties, diameter of $A_n < \varepsilon/2^n$ (for $n > 0$), $A_n \cap A_{n-1} \neq \varnothing$, and $p \in R[A_n]^-$ as follows: Set $A_0 = A$, $A_1 = U(u, \varepsilon/2^2)$. The step from A_{n-1} to A_n, $n > 1$, is to choose $\alpha_n \in \Omega$ such that for all $(x, y) \in R$, $\alpha_n^*(y) \subset R[U(x, \varepsilon/2^{n+1})]^-$. Then since $p \in R[A_{n-1}]^-$, $\alpha_n^*(p) \cap R[A_{n-1}] \neq \varnothing$, and so there exists $(u_n, v_n) \in R$ such that $u_n \in A_{n-1}$ and $v_n \in \alpha_n^*(p)$. Consequently, $p \in \alpha_n^*(v_n) \subset R[U(u_n, \varepsilon/2^{n+1})]^-$. Set $A_n = U(u_n, \varepsilon/2^{n+1})$. If $x_n \in A_n \cap A_{n-1}$ for $n \geqslant 1$, then (x_n) is a Cauchy sequence, and since X is complete, it converges to $x \in X$. Clearly $d(x, u) < \varepsilon$. We now show that $(x, p) \in R$. If W is any neighbourhood of x, then W contains some A_n, and so $p \in R[W]^-$. If V is any neighbourhood of p, then $V \cap R[W] \neq \varnothing$, and so there is an $s \in W$, $t \in V$ with $(s, t) \in R$. This shows that $R \cap (W \times V) \neq \varnothing$, and since R has a closed graph, $(x, p) \in R$. Thus $R^{-1}(p)\ \delta\ A$.

We now prove an analogue of the above for uniformly locally compact space [1, p. 214].

9. Corollary. *Let $(X, \mathfrak{U})$ be a uniformly locally compact space (i.e. there is a $U_0 \in \mathfrak{U}$ such that $U_0[x]$ is compact for all $x \in X$), (Y, Ω) be a Morita uniform space, $R \subset X \times Y$ be a graph-closed uniformly nearly open relation. Then R is uniformly open.*

Proof. It is sufficient to show that if $A \subset X$, $p \in R[A]^-$, then $R^{-1}(p)\ \delta\ A$. This will show that R is weakly open and $R[X]$ is closed. Thus by Theorem 7, it will follow that R is uniformly open.

Suppose then, that there exist $A \subset X$, $p \in R[A]^-$ such that $R^{-1}(p) \not\delta A$.

Let $U_0 \in \mathfrak{U}$ be such that $U_0[x]$ is compact for all x. Choose a symmetric $U \in \mathfrak{U}$ such that $U^2 \subset U_0$ and $R^{-1}(p) \cap U^2[A] = \varnothing$. Since R is uniformly nearly open, there is an $\alpha_1 \in \Omega$ such that for all $x \in X$, $\alpha_1^*(R(x)) \subset R[U[x]]^{-0}$. Since $p \in R[A]^- \subset \alpha_1^*(R[A])$, we can choose $x_1 \in A$ such that $p \in \alpha_1^*(R(x_1)) \subset R[U[x_1]]^-$. Now $U[x_1]^-$ is compact and does not meet $R^{-1}(p)$, i.e. $R \cap (U[x_1]^- \times \{p\}) = \varnothing$. For each $x \in U[x_1]^-$, we can find a $U_x \in \mathfrak{U}$, $\alpha_x \in \Omega$ such that $R \cap (U_x[x] \times \alpha_x^*(p)) = \varnothing$. Since $U[x_1]^-$ is compact, it can be covered by finitely many $U_x[x]$, and hence we can find an $\alpha \in \Omega$ such that $R \cap (U[x_1]^- \times \alpha^*(p)) = \varnothing$. But this contradicts $p \in R[U[x_1]]^-$.

We now show how our results are related to some recent results of Pettis [4].

10. THEOREM. *Suppose Y is Hausdorff, $R \subset X \times Y$ is a bijective relation with a metrically complete graph. If R is nearly open, then R is open.*

PROOF. Let P_Y (resp. P_X) be the projection from $X \times Y$ onto Y (resp. X), and P_{RY} (resp. P_{RX}) be the restriction of P_Y (resp. P_X) to R. Then since R is bijective, P_{RY} is bijective. Moreover, P_{RY} is continuous. Since $R = P_{RY} \circ P_{RX}^{-1}$, our result is proved if we show that P_{RY} is nearly open. Let V be open in R and $y_0 \in P_{RY}(V)$. There exist open sets U_X, U_Y in X, Y respectively, such that $(U_X \times U_Y) \cap R \subset V$, $R^{-1}(y_0) \in U_X$, $y_0 \in U_Y$. Then $y_0 \in R[U_X] \cap U_Y \subset P_{RY}(V)$. Since $R[U_x] \subset R[U_x]^{-0}$, $y_0 \in R[U_x]^{-0}$. But as U_Y is open, $U_Y \cap R[U_x]^{-0} = U_Y \cap (R[U_x] \cap U_Y)^{-0}$. Thus $y_0 \in P_{RY}(V)^{-0}$, and the result is proved.

Since a closed subset of a metrically complete space is metrically complete we get from the above,

11. COROLLARY. *If X and Y are metrically complete, $R \subset X \times Y$ is a bijective nearly open relation with a closed graph, then R is open.*

Theorem 10 and Corollary 11 are the relation form of (IV) and (V) in Pettis [4]. Under appropriate hypotheses on the spaces and on the function f: $X \to Y$, we can apply Theorem 10 to f^{-1} to yield (I) of [4], and Corollary 11 to f^{-1} to yield (II) in [4].

Finally, we answer a question of Pettis [5] by showing that Weston's result does not generalize to cocompact domain.

12. EXAMPLE. Let $X = \mathbf{R}$ with the topology having as subbase the family of all open intervals together with the rationals. Then X is a Hausdorff cocompact space [8, p. 10]. Let $Y = \mathbf{R}$ with the usual topology. Then the identity map from X to Y is a continuous nearly open bijection but is not open.

Added in revision. The referee suggested the following conjecture which includes our Theorem 5 as well as the one recently proved by T. Byczkowski and R. Pol (*On the closed graph and open mapping theorems*, Bull. Acad. Polon. Sci. **24** (1976), 723–726).

CONJECTURE. *Let X be regular and strongly countably complete, Y a topological space, $R \subset X \times Y$ a nearly open relation from X to Y, with $R[X]$ either open or closed in Y. Then R is open from X to Y if either* (i) *R is separating, or* (ii) *R has a closed graph, R is injective, and $R[A]$ is compact whenever A is compact.*

We are unable to prove the above conjecture. The following result is an improvement of our Theorem 5, but is independent of the result due to Byczkowski and Pol (loc cit). We observe that in our Theorem 3 we need δ to be only an R-proximity (see D. Harris, *Regular-closed spaces and proximities*, Pacific J. Math. **34** (1970), 675–685). For unfamiliar terms in the following see J. M. Aarts and D. J. Lutzer, *Completeness properties designed for recognizing Baire spaces*, Dissertationes Math. **116** (1974), 48 pp.

The proof of the following theorem is similar to that of T. Byczkowski and R. Pol (loc cit).

THEOREM. *Let X be a Rudin-complete Moore space, Y a topological space, $R \subset X \times Y$ a nearly open separating relation from X to Y with $R[X]$ either open or closed in Y. Then R is open from X to Y.*

PROOF. As in the proof of Theorem 5, it suffices to prove that R is weakly open when $R[X]$ is open. To this end, we first prove that for any open set U in X, $R[U]^- \subset R[U^-]$. Let $\{\mathcal{G}(n)\}$ be a nested development for X which is Rudin-complete. If $R[U]^- \not\subset R[U^-]$, then $R[U]^- \cap R[V] \neq \varnothing$, where $V = X \sim U^-$. Take $y_1 \in R[U]^- \cap R[V]$ and $x_1 \in R^{-1}(y_1) \cap V$. Let n_1 be such that

$$\mathrm{St}(x_1, \mathcal{G}(n_1)) = \bigcup \{G: x_1 \in G \in \mathcal{G}(n_1)\} \subset V,$$

and let $G_1 \in \mathcal{G}(n_1)$ be such that $x_1 \in G_1 \subset V$. Since $R[G_1]^-$ is a neighbourhood of y_1, $R[G_1]^- \cap R[U] \neq \varnothing$; hence there is a $z_1 \in R[G_1]^- \cap R[U]$ and $u_1 \in R^{-1}(z_1) \cap U$. Let m_1 be such that $\mathrm{St}(u_1, \mathcal{G}(m_1)) \subset U$ and $H_1 \in \mathcal{G}(m_1)$ be such that $u_1 \in H_1 \subset U$. Now choose inductively two sequences $\{n_i\}$ and $\{m_i\}$ such that

(i) $G_i \in \mathcal{G}(n_i)$, $H_i \in \mathcal{G}(m_i)$;

(ii) $R[H_i]^- \cap R[G_i] \neq \varnothing$; and

(iii) $G_{i+1}^- \subset G_i$, $H_{i+1}^- \subset H_i$.

((iii) follows from the fact that a Moore space is regular.)

Since X is Rudin-complete, $A_0 = \bigcap\{H_i: i \geqslant 1\}$ and $A_1 = \bigcap\{G_i: i \geqslant 1\}$ are nonvoid disjoint sets. Let $x_i \in A_i$ for $i = 0, 1$. Since R is separating, there are neighbourhoods N_i of x_i, $i = 0, 1$, such that $R[N_0]^- \cap R[N_1] = \varnothing$. Since $\{\mathcal{G}(n)\}$ is a development, there is an n such that $H_n \subset N_0$ and $G_n \subset N_1$ and so $R[H_n]^- \cap R[G_n] = \varnothing$, a contradiction. Thus $R[U]^- \subset R[U^-]$ for every open set U in X.

Now it follows that R is weakly open. Suppose $A \subset X$, $y \in R[A]^- \cap R[X]$ and $q = R^{-1}(y)$. If $q \not\in A^-$ then there is an open neighbourhood U of q such that $U^- \cap A^- = \varnothing$. Since R is injective, $R[U^-] \cap R[A^-] = \varnothing$. But

$R[U]^- \subset R[U^-]$, and hence $R[U]^- \cap R[A] = \varnothing$, a contradiction.

REMARKS.

(i) For a Moore space, Rudin-completeness is equivalent to subcompactness.

(ii) There is a completely regular Moore space X which is Rudin complete but not Čech complete.

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DEPARTMENT OF MATHEMATICAL SCIENCES, LAKEHEAD UNIVERSITY, THUNDER BAY 'P', ONTARIO, CANADA P7B 5E1