EXTENSION OF INVARIANT LINEAR FUNCTIONALS¹

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ABSTRACT. In a recent paper [4] we considered a semigroup of linear contractions or a group of linear isometries in a normed vector space, and obtained sufficient conditions for the orbits to lie on parallel closed hyperplanes. In the present note, we take a more general viewpoint and shall prove some theorems on extension of continuous linear functionals which are invariant under a group or a semigroup of continuous linear maps on a locally convex topological vector space. These more general results include those in [4] as direct consequences.

1. Preliminaries. We follow [1] for notation. For the convenience of the reader, we recall that the dual space of a locally convex topological vector space E is denoted by E' . The weak topology on E for the dual system $\langle E, E' \rangle$ is denoted by $\sigma(E, E')$, while $\sigma(E', E)$ is the weak topology on E' for the same dual system. $\tau(E', E)$ is the Mackey topology on E' for the dual system $\langle E, E' \rangle$. The polar in E' of a set C in E is denoted by C^0 , i.e., $C^0 = \{\varphi \in E' : \varphi(x) \leq 1 \text{ for } x \in C\}$. $\text{Int } K$ stands for the interior of a set K . The transpose of a continuous linear map u is denoted by ${}^t u$.

We begin with a lemma which will be needed in the proofs of Theorems 1 and 2. In the case when K is a convex cone, the lemma stated below becomes the well-known theorem of Krein [7, Theorem 1.1] (see also [1, Chapter II, p. 75], [10, p. 227]) on extension of positive linear functionals. This lemma can be derived from a purely algebraic lemma (i.e., involving no topological consideration) of Dixmier [2, p. 225], but we include here a direct proof for the sake of completeness.

LEMMA. *Let E be a real Hausdorff topological vector space. Let L be a linear subspace of E , and K a convex set in E such that $L \cap \text{Int } K \neq \emptyset$. If φ is a linear functional on L and α is a real number such that $\varphi(x) \leq \alpha$ for $x \in L \cap K$, then there exists a continuous linear functional ψ on E such that the restriction $\psi|_L$ of ψ on L is φ , and $\psi(x) \leq \alpha$ for $x \in K$.*

PROOF. If φ is the zero functional on L , then we may take the zero functional on E as ψ . We shall therefore assume that $\varphi \neq 0$. Let $M = \{x \in L : \varphi(x) = \alpha\}$. We claim that $M \cap \text{Int } K = \emptyset$. Suppose that $x_0 \in M \cap \text{Int } K$. Let V be a circled neighborhood of 0 in E such that $x_0 + V \subset K$. Then $V \cap L$ is a neighborhood of 0 in L , so φ is not identically zero on

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$V \cap L$. Since $V \cap L$ is circled, there is a point $y_0 \in V \cap L$ with $\varphi(y_0) > 0$. Then $x_0 + y_0 \in (x_0 + V) \cap L \subset K \cap L$, and $\varphi(x_0 + y_0) > \varphi(x_0) = \alpha$. This contradicts our hypothesis that $\varphi(x) \leq \alpha$ for $x \in K \cap L$. Thus $M \cap \text{Int } K = \emptyset$ and so, by the geometric form of Hahn-Banach theorem (see e.g. [1, Chapter II, p. 69]), there exist a continuous linear functional ω on E and a real number β such that $\omega(x) = \beta$ for $x \in M$, and $\omega(y) < \beta$ for $y \in \text{Int } K$.

By hypothesis, there is an $x_1 \in L \cap \text{Int } K$. Then $\omega(x_1) < \beta$, which together with the fact that $\omega(x) = \beta$ for $x \in M$, shows that ω is not identically zero on L . Let $x_2 \in M$ and $P = \{x \in L: \varphi(x) = 0\}$, $Q = \{x \in L: \omega(x) = 0\}$. Then $P = M - x_2 \subset Q$ and therefore $P = Q$. It follows that there is a nonzero real number λ such that $\varphi(x) = \lambda\omega(x)$ for $x \in L$. Then $\lambda\omega(x_1) = \varphi(x_1) < \alpha = \varphi(x_2) = \lambda\omega(x_2) = \lambda\beta$. But $\omega(x_1) < \beta$, so we have $\lambda > 0$ and $\alpha = \lambda\beta$. Define $\psi \in E'$ by $\psi = \lambda\omega$. Then $\psi|L = \varphi$. Since $\omega(y) < \beta$ for $y \in \text{Int } K$, we have $\psi(x) = \lambda\omega(x) \leq \lambda\beta = \alpha$ for $x \in K$.

2. Linear functionals invariant under a group. We first prove a theorem on extension of continuous linear functionals invariant under a group of continuous linear maps.

THEOREM 1. *Let E be a real Hausdorff locally convex topological vector space. Let L be a linear subspace of E , and K a convex set in E with $0 \in \text{Int } K$. Let G be a group of continuous linear maps of E onto itself such that*

$$(1) \quad u(L) = L, \quad u(K) = K \quad \text{for every } u \in G.$$

Let $\varphi \in L'$ be such that

$$(2) \quad \varphi \in (K \cap L)^0,$$

and

$$(3) \quad \varphi(u(x)) = \varphi(x) \quad \text{for } x \in L, u \in G.$$

If the orbit $G(x) = \{u(x): u \in G\}$ of each $x \in E$ is precompact, then there exists a $\psi \in E'$ such that

$$(4) \quad \psi \in K^0, \quad \psi|L = \varphi,$$

and

$$(5) \quad \psi(u(x)) = \psi(x) \quad \text{for } x \in E, u \in G.$$

PROOF. Let X be the set of all $\psi \in E'$ such that $\psi \in K^0$ and $\psi|L = \varphi$. By the Lemma above, X is nonempty. Since $0 \in \text{Int } K$, K^0 is $\sigma(E', E)$ -compact by Alaoglu-Bourbaki theorem. Consequently, X is a $\sigma(E', E)$ -compact set which is clearly convex.

Since $u(K) = K$ for $u \in G$, the relations $u \in G$ and $\psi \in X$ imply that

$$[{}^t u(\psi)](x) = \psi(u(x)) \leq 1 \quad \text{for } x \in K,$$

i.e., ${}^t u(\psi) \in K^0$. Similarly, because of $u(L) = L$ and (3) the relations $u \in G$ and $\psi \in X$ imply that

$$[{}^t u(\psi)](x) = \psi(u(x)) = \varphi(u(x)) = \varphi(x) \quad \text{for } x \in L,$$

i.e., $\langle u(\psi) | L = \varphi$. Hence $\langle u(X) \subset X$ for each $u \in G$.

We shall see that $\{u: u \in G\}$ is $\sigma(E', E)$ -equicontinuous on X . Let

$$V = \{\omega \in E': |\omega(x_i)| \leq \varepsilon \text{ for } 1 \leq i \leq n\}$$

be a given $\sigma(E', E)$ -neighborhood of 0 in E' , where $\{x_1, x_2, \dots, x_n\}$ is a finite set in E and $\varepsilon > 0$. The orbit $G(x_i)$ is precompact and $\varepsilon(K \cap -K)/3$ is a neighborhood of 0 in E , so for each $i = 1, 2, \dots, n$, there is a finite partition of G , $G = \bigcup_{j=1}^{m_i} G_j^{(i)}$, such that $u(x_i) - v(x_i)$ is in $\varepsilon(K \cap -K)/3$ whenever u, v are in the same set $G_j^{(i)}$ for some j . Let

$$G_{j_1 j_2, \dots, j_n} = G_{j_1}^{(1)} \cap G_{j_2}^{(2)} \cap \dots \cap G_{j_n}^{(n)},$$

where $1 \leq j_i \leq m_i$ ($1 \leq i \leq n$). Then

$$G = \bigcap_{i=1}^n \bigcup_{j=1}^{m_i} G_j^{(i)} = \bigcup_{(j_1, \dots, j_n)} G_{j_1 j_2, \dots, j_n}.$$

If u, v are in the same $G_{j_1 j_2, \dots, j_n}$ for some $(j_1, j_2, \dots, j_n)$, then

$$u(x_i) - v(x_i) \in \varepsilon(K \cap -K)/3 \text{ for all } i = 1, 2, \dots, n.$$

By picking an arbitrary element from each of the nonempty ones of the sets $G_{j_1 j_2, \dots, j_n}$, we obtain a finite subset $\{u_1, u_2, \dots, u_p\}$ of G with the property that for each $u \in G$, there is an index $k \in \{1, 2, \dots, p\}$ such that all the n relations

$$(6) \quad u(x_i) - u_k(x_i) \in \varepsilon(K \cap -K)/3 \quad (1 \leq i \leq n)$$

hold. Let W be the $\sigma(E', E)$ -neighborhood of 0 in E' defined by

$$W = \{\omega \in E': |\omega(u_k(x_i))| \leq \varepsilon/3 \text{ for } 1 \leq i \leq n, 1 \leq k \leq p\}.$$

Then the $\sigma(E', E)$ -equicontinuity of $\{u: u \in G\}$ on X will be proved by verifying the implication

$$(7) \quad \psi_1, \psi_2 \in X, \psi_1 - \psi_2 \in W \text{ and } u \in G \text{ imply } \langle u(\psi_1) - \langle u(\psi_2) \in V.$$

For a given $u \in G$, if we choose $k \in \{1, 2, \dots, p\}$ to satisfy (6) and if $\psi \in K^0$, then

$$|\psi[u(x_i) - u_k(x_i)]| \leq \varepsilon/3 \text{ for } 1 \leq i \leq n.$$

Therefore, for a given $u \in G$, if we choose $k \in \{1, 2, \dots, p\}$ to satisfy (6), then for any ψ_1, ψ_2 in X such that $\psi_1 - \psi_2 \in W$, we have

$$\begin{aligned} |[\langle u(\psi_1) - \langle u(\psi_2)](x_i)| &= |(\psi_1 - \psi_2)(u(x_i))| \\ &\leq |\psi_1[u(x_i) - u_k(x_i)]| + |\psi_2[u(x_i) - u_k(x_i)]| + |(\psi_1 - \psi_2)[u_k(x_i)]| \\ &\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \end{aligned}$$

for $i = 1, 2, \dots, n$; i.e., $\langle u(\psi_1) - \langle u(\psi_2) \in V$. This verifies (7) and therefore the $\sigma(E', E)$ -equicontinuity of $\{u: u \in G\}$ on X .

We have seen that X is a nonempty, $\sigma(E', E)$ -compact convex set and that the restrictions $\{u|X: u \in G\}$ of $\{u: u \in G\}$ on X form a group of

$\sigma(E', E)$ -equicontinuous affine maps of X onto itself. By a well-known fixed-point theorem of Kakutani [5], [3, p. 457], there exists a $\psi \in X$ such that $u(\psi) = \psi$ for all $u \in G$. This ψ has properties (4) and (5).

COROLLARY 1. *In a real Hausdorff locally convex space E , let K be a convex set with $0 \in \text{Int } K$ and z_0 a boundary point of K . Let G be a group of continuous linear maps of E onto itself such that $u(K) = K$ and $u(z_0) = z_0$ for every $u \in G$. If the orbit $G(x) = \{u(x) : u \in G\}$ of each $x \in E$ is precompact, then there exists a continuous linear functional ψ on E such that $\psi \in K^0$, $\psi(z_0) = 1$ and $\psi(u(x)) = \psi(x)$ for all $x \in E$, $u \in G$.*

PROOF. Let L be the one-dimensional linear subspace spanned by z_0 . Since $0 \in \text{Int } K$ and z_0 is a boundary point of K , every real number λ satisfying $\lambda z_0 \in K$ must be ≤ 1 . Let $\varphi \in L'$ be defined by $\varphi(\lambda z_0) = \lambda$ for every real number λ . Then $\varphi(\lambda z_0) = \lambda \leq 1$ for $\lambda z_0 \in K$, so $\varphi \in (K \cap L)^0$. Then the result follows immediately from Theorem 1.

In the special case when E is a normed vector space, K is the unit ball in E and G is a group of linear isometries, Corollary 1 becomes Theorem 3 in [4]. As was pointed out in [4], this result includes as special cases the mean value theorem of von Neumann for almost periodic functions as well as the existence of Haar integral on a compact group (or more generally, on the space of left cosets of a compact group).

3. Linear functionals invariant under a semigroup. Let E be a Hausdorff locally convex topological vector space, and let S be a semigroup of continuous linear maps of E into itself. As in [4], we shall say that a continuous linear functional φ on E is a *direction of deflation* for S , if φ is not the zero functional and φ has the following property (D):

(D) For every circled, convex, $\sigma(E, E')$ -compact set C in E and any $\varepsilon > 0$, there is a $u \in S$ such that

$$(8) \quad u(C) \subset \{x \in E : |\varphi(x)| \leq \varepsilon\}.$$

Now, for a circled, convex, $\sigma(E, E')$ -compact set C in E and $\varepsilon > 0$, the set C/ε is also circled, convex and $\sigma(E, E')$ -compact. Therefore property (D) of φ is equivalent to the following property: for every circled, convex, $\sigma(E, E')$ -compact C in E , there is a $u \in S$ such that $u(C) \subset \{x \in E : |\varphi(x)| \leq 1\}$. We shall say that the semigroup S is *nondeflating*, if S has no direction of deflation.

THEOREM 2. *Let E be a real Hausdorff locally convex topological vector space. Let L be a linear subspace of E , K a convex set in E , and $x_0 \in L \cap \text{Int } K$. Let S be a nondeflating semigroup of continuous linear maps of E into itself such that*

$$(9) \quad u(L) \subset L \quad \text{and} \quad u(K - x_0) \subset K - x_0 \quad \text{for } u \in S.$$

If $\varphi \in L'$ and a real number α are such that

$$(10) \quad \varphi(u(x)) = \varphi(x) \quad \text{for } x \in L \text{ and } u \in S,$$

$$(11) \quad \varphi(x) \leq \alpha \quad \text{for } x \in L \cap K,$$

then there exists $\psi \in E'$ such that

$$(12) \quad \psi|L = \varphi,$$

$$(13) \quad \psi(u(x)) = \psi(x) \quad \text{for } x \in E \text{ and } u \in S,$$

$$(14) \quad \psi(x) \leq \alpha \quad \text{for } x \in K.$$

PROOF. We shall assume that φ is not the zero functional on L , for otherwise we can take the zero functional on E as ψ . From $\varphi \neq 0$ and (11) we must have $\varphi(x) < \alpha$ for $x \in L \cap \text{Int } K$. In particular, $\varphi(x_0) < \alpha$. Let $\varphi_0 = \varphi/(\alpha - \varphi(x_0))$ and $K_0 = K - x_0$. Then, since $L \cap K_0 = (L \cap K) - x_0$, (11) implies that $\varphi_0(x) \leq 1$ for $x \in L \cap K_0$; i.e., $\varphi_0 \in (L \cap K_0)^0$. Let X be the set of all $\omega \in E'$ such that $\omega \in K_0^0$ and $\omega|L = \varphi_0$. Since $0 \in L \cap \text{Int } K_0$, X is nonempty by the Lemma. Also since $0 \in \text{Int } K_0$, K_0^0 and therefore X are $\sigma(E', E)$ -compact. X is clearly convex. For $u \in S$ and $\omega \in X$, we have

$$[{}^t u(\omega)](x) = \omega(u(x)) \leq 1 \quad \text{for } x \in K_0$$

(because $u(K_0) \subset K_0$), and

$$[{}^t u(\omega)](x) = \omega(u(x)) = \varphi_0(u(x)) = \varphi_0(x) \quad \text{for } x \in L$$

(because $u(L) \subset L$ and (10)). Hence ${}^t u(X) \subset X$ for $u \in S$.

Consider now any two distinct elements ω_1, ω_2 of X . Since S is nondeflating, we can find a circled, convex, $\sigma(E, E')$ -compact set C in E and an $\varepsilon > 0$ such that $u(C) \not\subset \{y \in E: |(\omega_1 - \omega_2)(y)| \leq \varepsilon\}$ for every $u \in S$. Then for every $u \in S$, there is an $x \in C$ such that

$$|[{}^t u(\omega_1 - \omega_2)](x)| = |(\omega_1 - \omega_2)(u(x))| > \varepsilon.$$

Thus ${}^t u(\omega_1 - \omega_2) \notin \varepsilon C^0$ for every $u \in S$. In other words, εC^0 is disjoint from the set $\{{}^t u(\omega_1 - \omega_2): u \in S\}$. Since C is circled, convex and $\sigma(E, E')$ -compact, εC^0 is a $\tau(E', E)$ -neighborhood of 0 in E' . Therefore 0 is not in the $\tau(E', E)$ -closure of the set $\{{}^t u(\omega_1 - \omega_2): u \in S\}$.

The restrictions $\{{}^t u|X: u \in S\}$ on X of $\{{}^t u: u \in S\}$ form a semigroup of $\sigma(E', E)$ -continuous affine maps of the $\sigma(E', E)$ -compact convex set X into itself. This semigroup has the property that for any two distinct elements ω_1, ω_2 of X , 0 is not in the $\tau(E', E)$ -closure of $\{{}^t u(\omega_1 - \omega_2): u \in S\}$. By Ryll-Nardzewski's fixed point theorem [9], [8], there exists a $\psi_0 \in X$ such that ${}^t u(\psi_0) = \psi_0$ for all $u \in S$. Let $\psi = (\alpha - \varphi(x_0))\psi_0$. Then (12) is seen from

$$\psi(x) = (\alpha - \varphi(x_0))\psi_0(x) = (\alpha - \varphi(x_0))\varphi_0(x) = \varphi(x) \quad \text{for } x \in L.$$

(13) follows immediately from ${}^t u(\psi_0) = \psi_0$ for all $u \in S$. If $x \in K$, we can write $x = y + x_0$ with $y \in K_0$ and

$$\begin{aligned} \psi(x) &= (\alpha - \varphi(x_0))[\psi_0(y) + \psi_0(x_0)] \leq (\alpha - \varphi(x_0))(1 + \varphi_0(x_0)) \\ &= \alpha - \varphi(x_0) + \varphi(x_0) = \alpha, \end{aligned}$$

which verifies (14). The proof is complete.

COROLLARY 2. *In a real Hausdorff locally convex space E , let K be a convex set and $x_0 \in \text{Int } K$. Let S be a nondeflating semigroup of continuous linear maps of E into itself such that $u(K) \subset K$ and $u(x_0) = x_0$ for every $u \in S$. If $\lambda x_0 \notin K$ for some real number λ , then there exist a nonzero continuous linear functional ψ on E and a real number α such that $\psi(x) \leq \alpha$ for $x \in K$; and $\psi(u(x)) = \psi(x)$ for all $x \in E$ and $u \in S$.*

PROOF. Denote by L the one-dimensional linear subspace spanned by x_0 . Since $\lambda x_0 \notin L \cap K$, we can find a nonzero $\varphi \in L'$ and a real number α such that $\varphi(x) \leq \alpha$ for $x \in L \cap K$ and $\varphi(\lambda x_0) \geq \alpha$. Then by Theorem 2, φ can be extended to a $\psi \in E'$ with the desired properties.

COROLLARY 3. *In a real Hausdorff locally convex space E , let $K \neq E$ be a convex cone (with vertex at 0) and $x_0 \in \text{Int } K$. If S is a nondeflating semigroup of continuous linear maps of E into itself such that $u(K) \subset K$ and $u(x_0) = x_0$ for every $u \in S$, then there is a nonzero continuous linear functional ψ on E such that $\psi(x) \geq 0$ for $x \in K$; and $\psi(u(x)) = \psi(x)$ for $x \in E$ and $u \in S$.*

PROOF. Observe first that $-x_0 \notin K$, for otherwise $-x_0 \in K$ combined with $x_0 \in \text{Int } K$ would imply $0 \in \text{Int } K$ and then the convex cone K would coincide with E . By Corollary 2, there exist a nonzero $\psi \in E'$ and a real number α such that $\psi(x) \geq \alpha$ for $x \in K$ and $\psi(u(x)) = \psi(x)$ for $x \in E$ and $u \in S$. Since K is a cone and $\psi(x) \geq \alpha$ for $x \in K$, we must have $\psi(x) \geq 0$ for $x \in K$.

Corollary 3 is a natural generalization of Krein's classical theorem [7, Theorem 1.1] on extension of positive linear functionals. On the other hand, Theorem 1 in [4] on the orbits of a nondeflating semigroup of linear contractions in a normed vector space is easily seen to be another specialization of the above Theorem 2.

4. Groups with relatively $\sigma(E, E')$ -compact orbits. For a group of continuous linear maps, we apply Theorem 2 to prove the following result which is a variant of Theorem 1.

THEOREM 3. *Let E be a real Hausdorff locally convex space which is quasi-complete for the Mackey topology $\tau(E, E')$. Let L be a linear subspace of E , K a convex set in E and $x_0 \in L \cap \text{Int } K$. Let G be a group of continuous linear maps of E onto itself such that $u(L) = L$ and $u(K - x_0) = K - x_0$ for every $u \in G$. Let $\varphi \in L'$ and a real number α be such that $\varphi(x) \leq \alpha$ for $x \in L \cap K$, and $\varphi(u(x)) = \varphi(x)$ for $x \in L$, $u \in G$. If the orbit $G(x) = \{u(x): u \in G\}$ of each $x \in E$ is relatively compact for the weak topology $\sigma(E, E')$, then φ can be extended to a $\psi \in E'$ such that $\psi(x) \leq \alpha$ for $x \in K$, and $\psi(u(x)) = \psi(x)$ for $x \in E$, $u \in G$.*

PROOF. By Theorem 2, it suffices to show that G is nondeflating. For each $x \in E$ let C_x be the closed, circled, convex hull of the orbit $G(x)$ of x . Since $G(x)$ is relatively $\sigma(E, E')$ -compact and E is quasi-complete for the Mackey

topology $\tau(E, E')$, C_x is $\sigma(E, E')$ -compact by Krein's theorem [6, §24.5], [10, p. 189]. It is easy to see that $u(C_x) \subset C_x$ and therefore, by group property, $u(C_x) = C_x$ for every $u \in G$ and $x \in E$. If $\varphi \in E'$ has property (D), then we must have $C_x \subset \{y \in E: |\varphi(y)| \leq \varepsilon\}$ for every $x \in E$ and every $\varepsilon > 0$. Thus the only $\varphi \in E'$ having property (D) is the zero functional. This proves that G is nondeflating.

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