WHEN IS THE FREE PRODUCT OF LATTICES COMPLETE?

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ABSTRACT. Yu. I. Sorkin proved that, up to isomorphism, there are three finite lattices that can be represented as a free product of two lattices. In this note we prove that, up to isomorphism, there are five complete lattices that can be so represented.

1. Let $A * B$ denote the free product of the lattices A and B , and let C_n denote the n -element chain.

THEOREM. *A free product $L = A * B$ of two lattices is complete iff L is isomorphic to one of the following five lattices: $C_1 * C_n$, $n = 1, 2, 3, 4$, and $C_2 * C_2$. In fact, up to isomorphism, $\{A, B\}$ is one of $\{C_1, C_n\}$, $n = 1, 2, 3, 4$, or $A \cong B \cong C_2$.*

This result generalizes a theorem of Yu. I. Sorkin [5], describing when $L = A * B$ is finite; we get his result by deleting the infinite lattices $C_1 * C_4$ and $C_2 * C_2$ from the above list.

2. The “if” part of the Theorem is implicit in a paper of H. L. Rolf [4] where $C_1 * C_4$ and $C_2 * C_2$ are described by (generalized) diagrams. (These diagrams are reproduced in G. Grätzer [1].) A simple inspection of these diagrams shows that $C_1 * C_4$ and $C_2 * C_2$ are (infinite and) complete. $C_1 * C_n$, $n = 1, 2, 3$, are finite, hence complete, by Yu. I. Sorkin [5]. (For their diagrams, see also G. Grätzer [1].)

3. We need some simple facts about isotone retracts of lattices.

Let L be a lattice and let K be a sublattice of L . A map $\varphi: L \rightarrow K$ is a *retraction* (and K is a *retract* of L) iff $x\varphi = x$ for all $x \in K$. If φ is also an isotone map, then φ is called an *isotone retraction* and K an *isotone retract* of L .

The next two results are trivial and well known.

LEMMA 1. *An isotone retract of a complete lattice is complete.*

LEMMA 2. *Every finite sublattice of a lattice is an isotone retract.*

4. Let L be a free product of the lattices A and B ; we assume that A and B are sublattices of L . Let L^b denote L with new bounds 0^b and 1^b added to L .

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For $x \in L$, $(x] \cap A$ is principal or $\emptyset$; we define $x_A \in A$ by $(x_A] = (x] \cap A$ in the first case and $x_A = 0^b$ in the second case (see G. Grätzer, H. Lakser and C. R. Platt [2] and also G. Grätzer [1]). We define x_B similarly, and x^A , x^B dually.

LEMMA 3. Let $L = A * B$. Let A' be an isotone retract of A and let B' be an isotone retract of B . Let L' denote the sublattice of L generated by $A' \cup B'$. Then L' is an isotone retract of L .

REMARK. By a result of B. Jónsson [3], L' is a free product of A' and B' .

PROOF. Let $\alpha: A \rightarrow A'$ and $\beta: B \rightarrow B'$ be isotone retractions. We define a map $\lambda: L \rightarrow L'$:

For $x \in A$, $x\lambda = x\alpha$ and for $x \in B$, $x\lambda = x\beta$; if $x = y \vee z$, then $x\lambda = y\lambda \vee z\lambda \vee x_A \vee x_B$; if $x = y \wedge z$, define $x\lambda$ dually.

It was proved in G. Grätzer, H. Lakser and C. R. Platt [2], that λ is well defined on L and λ is isotone. A simple induction shows that $x\lambda \in L'$ for all $x \in L$ and that $x\lambda = x$ for all $x \in L'$. Thus λ establishes that L' is an isotone retract of L .

5. We need one more observation.

LEMMA 4. Let L be a countable lattice. If L contains $F(3)$ (the free lattice on three generators) as subposet, then L is not complete.

PROOF. Since $F(3) \supseteq F(\aleph_0)$ (Ph. M. Whitman [6]) and $F(\aleph_0)$ contains the chain of rationals, if L were complete, then L would contain the chain of real numbers, contradicting $|L| = \aleph_0$.

6. Now we prove the "only if" part of the Theorem. Let us assume that $L = A * B$ is complete.

If A is not a chain, then $A \supseteq F(2)$. By Lemma 2, $F(2)$ is an isotone retract of A , hence by Lemma 3 and the Remark, $F(2) * F(1) \cong F(3)$ is an isotone retract of L . By Lemma 1, $F(3)$ would be complete, contradicting Lemma 4.

Hence we can assume that both A and B are chains. It was shown in H. L. Rolf [4], that $F(3)$ is isomorphic to a sublattice of $C_1 * C_5$ and also of $C_2 * C_3$. Now if $|A| \geq 5$, then C_5 is an isotone retract of A , hence by Lemma 3, $C_1 * C_5$ is complete, contradicting Lemma 4. Thus $|A| \leq 4$ and $|B| \leq 4$. Now if $|A| \geq 3$ and $|B| \geq 2$ (or symmetrically), then we proceed as above. Thus either $A \cong B \cong C_2$ or A is a singleton and B has at most three elements or symmetrically. This completes the proof of the Theorem.

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