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SUCCESSIVE DERIVATIVES OF ENTIRE FUNCTIONS

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ABSTRACT. We show that if f is a real entire function which has, along with each of its derivatives, only real nonpositive zeros, then either $f(z) = ce^{\sigma z}$, c and σ real constants, or

$$f(z) = cz^m e^{\sigma z} \prod_n \left(1 + \frac{z}{|z_n|}\right)$$

where $\sigma > 0$ and $\Sigma_n |z_n|^{-1} < \infty$.

This note concerns a subclass of the class $\mathcal{U}_0$ of entire functions g of the form

$$g(z) = cz^m e^{-\gamma z^2 + \sigma z} \prod_n \left(1 - \frac{z}{z_n}\right) e^{z/z_n} \tag{1}$$

where c is a constant, $\gamma \geqslant 0$, σ and the z_n are real, and $\Sigma_n |z_n|^{-2} < \infty$. The subclass of $\mathcal{U}_0$ that we will be interested in is the class $\mathcal{U}$ of entire functions f of the form

$$f(z) = cz^m e^{\sigma z} \prod_n \left(1 + \frac{z}{|z_n|}\right) \tag{2}$$

where $\sigma \geqslant 0$ and $\Sigma_n |z_n|^{-1} < \infty$.

The class $\mathcal{U}_0$ (often called the Laguerre-Pólya class) and its subclass $\mathcal{U}$ are of special interest since classical theorems of Laguerre [3] and Pólya [4] assert that $f \in \mathcal{U}_0(\mathcal{U})$ if and only if f can be uniformly approximated on discs about the origin by a sequence of polynomials with only real (real nonpositive) zeros. A corollary of their results is: $f \in \mathcal{U}_0(\mathcal{U})$ implies $f^{(n)} \in \mathcal{U}_0(\mathcal{U})$, $n = 1, 2, \ldots$; in particular, $f \in \mathcal{U}_0(\mathcal{U})$ implies $f^{(n)}$ has only real (real nonpositive) zeros $n = 1, 2, \ldots$. In 1915, Pólya [6] (see also [5]) asked whether the following converse assertion holds: If a (constant multiple of a) real entire function f (i.e., z real implies $f(z)$ real) and each of its derivatives $f^{(n)}$, $n = 1, 2, \ldots$ have only real (real nonpositive) zeros is $f \in \mathcal{U}_0(\mathcal{U})$?

Pólya showed [5], [6] that if $f(z) = P(z)e^{Q(z)}$ where P and Q are polynomials, then the answer to this question is affirmative. Recently the authors

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established [1], [2] the full conjecture for $\mathfrak{U}_0$ with the following

THEOREM A. *Let f be a (constant multiple of a) real entire function. If f, f', and f'' have only real zeros, then $f \in \mathfrak{U}_0$.*

In this note we use this result to establish the conjecture for $\mathfrak{U}$ with the

THEOREM B. *Let f be a (constant multiple of a) real entire function. If f and $f^{(n)}$ have only real nonpositive zeros $n = 1, 2, \ldots,$ then either $f(z) = ce^{\sigma z}$, σ real, or $f \in \mathfrak{U}$.*

Before proving Theorem B, we remark that its hypotheses cannot be weakened by requiring only that f and $f^{(n)}$, $n = 1, 2, \ldots, N$, for some N have only real nonpositive zeros – see Remark 2 below.

PROOF OF THEOREM B. From Theorem A we see that

$$f(z) = cz^m e^{-\gamma z^2 + \sigma z} \prod(z) \tag{3}$$

where c is a constant which, without loss of generality, we take to be 1, and where $\prod(z)$ is a canonical product of genus $\leqslant 1$ with only negative zeros. Thus, it only remains to show that if $f(z) \neq ce^{\sigma z}$, then (i) $\gamma = 0$, (ii) $\prod$ is of genus 0, and (iii) $\sigma \geqslant 0$. To do this we first observe that there is no loss in generality in assuming that f and $f^{(k)}$, $k = 1, 2, \ldots,$ have only negative zeros. Indeed, if $f^{(k)}$ has a zero at the origin for some k ($f^{(0)} = f$), consider $g(z) = f(z + \varepsilon)$ where $\varepsilon > 0$. Then g and $g^{(k)}$, $k = 1, 2, \ldots,$ have only negative zeros, and if this implies $g \in \mathfrak{U}$, then clearly $f \in \mathfrak{U}$. Proceeding under this assumption then, we set $m = 0$ in (3) and next observe that since $f \in \mathfrak{U}_0, f(z) \neq ce^{\sigma z}$, either f or $f^{(k)}$, $k = 1, 2, \ldots,$ must have some (negative) zeros. Also, if

$$f(z) = 1 + \sum_{n=1} c_n \frac{z^n}{n!} \tag{4}$$

then

$$c_n = f^{(n)}(0) \neq 0, \qquad n = 1, 2, \ldots; \tag{5}$$

moreover, it cannot be the case that $c_n c_{n+1} < 0$ for $n = 0, 1, 2, \ldots$ for then

$$(-1)^k f^{(k)}(-x) = \sum_{n=0} (-1)^{n+k} c_{n+k} \frac{x^n}{n!} > 0 \quad \text{for } x = \operatorname{Re} z > 0$$

and $f^{(k)}$ cannot have any negative zeros $k = 0, 1, 2, \ldots$ ($f^{(0)} = f$). Thus, there is at least one nonnegative integer n for which $c_n c_{n+1} > 0$; choose such an n and denote it by k, so that

$$c_k c_{k+1} > 0. \tag{6}$$

Since, as was pointed out above, $f \in \mathfrak{U}_0$ implies $f^{(k)} \in \mathfrak{U}_0$, we write

$$f^{(k)}(z) = c_k e^{-\gamma_k z^2 + \sigma_k z} \prod_k(z) \tag{7}$$

where $\gamma_k \geqslant 0$, σ_k is a real constant, and $\prod_k$ is a canonical product, finite or infinite, of genus $\leqslant 1$ with only negative zeros. Taking the logarithmic derivative of (7), we have

$$\frac{f^{(k+1)}}{f^{(k)}}(z) = -2\gamma_k z + \sigma_k + \frac{\Pi_k'}{\Pi_k}(z). \tag{8}$$

We now need the following elementary facts about the growth of canonical products of finite genus with only negative zeros: If $P(z)$ is a canonical product of genus q with only negative zeros $\{-a_n\}$, $a_n > 0$, then

$$\frac{P'}{P}(z) = (-1)^q z^q \sum_n \frac{1}{a_n^q(z + a_n)}, \qquad z \neq -a_n,$$

from which it follows readily that for $x = \operatorname{Re} z$

$$\left|\frac{P'}{P}(x)\right| = o(x^q) \quad (x \to +\infty) \tag{9}$$

and

$$\lim_{x \to +\infty} \frac{P'}{P}(x) = -\infty \quad (q \text{ odd}). \tag{10}$$

We will now establish (i)–(iii). Since, as is easily verified, $\gamma_k = \gamma$ and genus Π_k = genus Π,[3] (8)–(10) imply that if either $\gamma > 0$ or Π is of genus 1 then

$$\lim_{x \to +\infty} \frac{f^{(k+1)}}{f^{(k)}}(x) = -\infty \quad (x = \operatorname{Re} z). \tag{11}$$

Further, (5) and (6) imply that

$$\frac{f^{(k+1)}}{f^{(k)}}(0) > 0. \tag{12}$$

It now follows from (11) and (12) that $f^{(k+1)}$ has a positive zero, which is a contradiction. Thus $\gamma = \gamma_k = 0$, Π and Π_k are of genus 0, and, by (3) (with $m = 0$) and (7)

$$f(z) = e^{\sigma z}\Pi(z), \qquad f^{(k)}(z) = c_k e^{\sigma_k z}\Pi_k(z),$$

where, as is easily verified, $\sigma = \sigma_k$. It then follows from (8) (with $\gamma_k = 0$) and (9) (with $q = 0$) that if $\sigma = \sigma_k < 0$, then

$$\lim_{x \to +\infty} \frac{f^{(k+1)}}{f^{(k)}}(x) = \sigma_k < 0 \quad (x = \operatorname{Re} z). \tag{13}$$

Reasoning as above, (13) yields a contradiction. Thus $\sigma \geqslant 0$, and the proof of Theorem B is complete.

REMARK 1. If we alter the hypothesis of Theorem B by dropping the condition that f be real but requiring instead that f be of finite order, we can then use Theorem 2 of [1] and the above proof to conclude that either $f(z) = ae^{bz}$, a and b constants, or $f \in \mathcal{U}$.

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[3] If $f \in \mathcal{U}_0$, then f' has only real zeros and $(f'/f)'(x) < 0$. Thus f' has exactly one zero, a simple one, between two consecutive zeros of f, so that the genus of π_1 = genus of π. A simple growth argument shows that $\gamma_1 = \gamma$. Introduction completes the argument. In the case $\gamma = 0$, cf. E. Borel, *Leçons sur les fonctions entières*, Paris, 1921, p. 32.

REMARK 2. Theorem B is false if it is only required that f and $f^{(n)}$, $n = 1, 2, \ldots, N$, for some N have only real nonpositive zeros. To see this, let f be of the form (3) with only real nonpositive zeros, $f \notin \mathfrak{U}$. By Laguerre's inequality, $[f^{(n)}(z)/f^{(n-1)}(z)]' < 0$ for $z \neq$ zero of $f^{(n-1)}$; thus, if we denote by z_{1N} the largest zero of $f^{(N)}$, it is clear that all zeros of f and $f^{(n)}$ lie in $(-\infty, z_{1N}]$, $n = 1, 2, \ldots, N$. If $z_{1N} > 0$, let $g(z) = f(z + z_{1N})$ and if $z_{1N} \leq 0$, let $g(z) = f(z)$. In either case g and $g^{(n)}$, $n = 1, 2, \ldots, N$, have only real nonpositive zeros; however, $g \notin \mathfrak{U}$.

ADDED IN PROOF. Part of our proof involved showing that if $f \in \mathfrak{U}_0$ and $f^{(k)}$ has only real nonpositive zeros for all k, then $\gamma = 0$ in (1). This also follows from a theorem of Edrei [Scripta Math. **22** (1956), Theorem 1] which readily implies the following

THEOREM C. *Let $F(z) = e^{-\gamma z^2}G(z)$, $\gamma > 0$ and $G \in \mathfrak{U}_0$ of genus ≤ 1. Then the zeros of the successive derivatives of f are everywhere dense on the real axis.*

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