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THE ISING MODEL LIMIT OF φ^4 LATTICE FIELDS

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ABSTRACT. We show that the $\lambda \to \infty$ limit of $\lambda\phi^4$ lattice fields is an Ising model.

I. **Introduction**. One of the basic problems of constructive quantum field theory concerns the existence and nontriviality of ϕ^4 quantum fields in 4 space-time dimensions. One approach is to first study $\lambda\phi^4$ fields on a lattice, and then let the lattice spacing shrink to zero [4]. The case of $\lambda = 0$ corresponds to the trivial free field. In this paper we prove that $\lambda = \infty$ corresponds to the Ising model. This result is easy to see in a finite volume, based on

$$\lim_{\lambda\to\infty} \exp\left(-\lambda(x^2-1)^2\right) dx \Big/ \int \exp\left(-\lambda(y^2-1)^2\right) dy \to \frac{\delta(x+1)}{2} + \frac{\delta(x-1)}{2}.$$

Our contribution is to establish this result for the infinite volume lattice fields. This result indicates that the nontriviality of $\lambda\phi^4$ fields should depend on the nontriviality of the scaling limit, as the lattice spacing tends to zero, of the Ising model [4].

Related to our result is the fact that the scaling limit of the x^4 anharmonic oscillator is the one-dimensional continuum Ising model (Poisson process) [3]. In an entirely different direction, we note that the ϕ^4 field can be approximated by ferromagnetic Ising models [5].

II. **Ising and ϕ^4 models**. Ising models are defined in terms of probability measures on $\Omega = \{-1, 1\}^{\mathbf{Z}^d}$. Let us call a configuration on $L \subseteq \mathbf{Z}^d$ a cylinder set which is determined by specifying the values of the coordinates σ_i, $i \in L$. Given a configuration B on $\partial L = \{i | \text{dist}(i, L) = 1\}$ (boundary conditions), let $P_{L,B}(\cdot)$ be the probability which is concentrated on the cylinder sets with base $\{-1, 1\}^L$, and such that

$$(1) \qquad P_{L,B}(A) = \exp(-H_{L,B}(A))/\text{Normalization}$$

for all configurations A on L. Here

$$H_{L,B}(A) = -\beta \sum_{\substack{|i-j|=1 \\ i\in L}} \sigma_i\sigma_j|_{A\cap B} - h\sum_{i\in L} \sigma_i|_A,$$

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and the inverse temperature β and magnetic field h are fixed throughout this section. Later, we will use the probability $P_L(\cdot)$ obtained with

$$H_L(A) = -\beta \sum_{\substack{|i-j|=1 \\ i,j\in L}} \sigma_i\sigma_j|_A - h\sum_{i\in L}\sigma_i|_A$$

(free boundary conditions).

Any probability obtained as a weak limit of $P_{L,B(L)}(\cdot)$, $L\uparrow \mathbf{Z}^d$, for some choice of $B(L)$ is called an Ising model probability. Among translation invariant probabilities on Ω, the Ising model probabilities are those satisfying

$$(2) \qquad P(A|B) = \exp(-H_{L,\partial B}(A))/\text{Normalization}$$

for all finite configurations A on L, B on L' with $L' \cap L = \varnothing$, $L' \supseteq \partial L$. Here ∂B is the configuration on ∂L with $\sigma_i|_{\partial B} = \sigma_i|_B$, $i \in \partial L$. If Π_K flips the spins σ_i indexed by $i \in K$, (2) is equivalent to

$$(3) \qquad P(\Pi_K A|B) = \exp(-H_{L,\partial B}(\Pi_K A) + H_{L,\partial B}(A))P(A|B).$$

(2) and (3) are the *DLR* equations [6], [7], [8]. ϕ^4 lattice fields are defined in terms of probability measures on $\mathbf{R}^{\mathbf{Z}^d}$. Let $\mu_L^\nu(\cdot)$ be the probability measure which is concentrated on the cylinder sets with base $\mathbf{R}^L$, and such that

$$\mu_L^\nu(A) = \int_A \frac{\exp\Big(\mathcal{H}_L(x_1,\ldots,x_{|L|}) - \Sigma_{i\in L}(x_i^2-1)^2/\nu\Big)dx_1,\ldots,dx_{|L|}}{\text{Normalization}}.$$

Here

$$\mathcal{H}_L(x_1,\ldots,x_{|L|}) = -\beta \sum_{\substack{|i-j|=1 \\ i,j\in L}} x_ix_j + \big(d\beta + m_0^2\big)\sum_{i\in L} x_i^2 - h\sum_{i\in L} x_i.$$

Later, we will use the probability measure $\mu_{L,P}^\nu(\cdot)$ obtained with

$$\mathcal{H}_{L,P}(x_1,\ldots,x_{|L|}) = \frac{\beta}{2}\sum_{\substack{|i-j|=1 \\ i,j\in L}} (x_i - x_j)^2 + m_0^2\sum_{i\in L} x_i^2 - h\sum_{i\in L} x_i,$$

where $|i-j|_T$ is the distance from i to j on the torus L. The weak limit μ^ν of μ_L^ν as $L\uparrow\mathbf{Z}^d$ exists and is translation invariant [12, pp. 289,293]. It is easy to see that $\mu_L^\nu(\cdot) \to P_L(\cdot)$ as $\nu\to 0$, since

$$\frac{\exp\Big(-(x^2-1)^2/\nu\Big)\,dx}{\text{Normalization}} \to \frac{\delta(x+1)}{2} + \frac{\delta(x-1)}{2}.$$

In the next section we prove an analogous statement for the infinite volume measures, μ^ν.

III. The $\nu \to 0$ limit.

THEOREM 1. *Let $\nu_j \to 0$. Then $\{\mu^{\nu_j}\}$ is weakly compact, and every limit point is a translation invariant Ising model probability.*

COROLLARY 1. *If $h \neq 0$, or if β is sufficiently small, μ^ν converges as $\nu\to 0$ to* *the unique translation invariant Ising model probability.*

COROLLARY 2. *If $d = 2$, μ^ν converges as $\nu \to 0$ to the unique translation invariant Ising model probability P with $P(\sigma_i = 1) = \frac{1}{2}$.*

Corollary 1 follows from our theorem and the fact that if $h \neq 0$ or if $\beta > 0$ is sufficiently small, there is a unique translation invariant Ising model [6], [9]. Similarly, Corollary 2 follows from the fact that in two dimensions the translation invariant Ising models are determined by $P(\sigma_i = 1)$ [10].

PROOF OF THEOREM 1. The proof proceeds in three steps. We first show that

$$\lim_{\nu \to 0} \int \exp\left(\lambda\left(x_i^2 - 1\right)\right) d\mu^\nu \leqslant e^{2d\beta} \tag{4}$$

independently of $\lambda \geqslant 0$. This implies that $\{\mu^\nu\}$ is weakly compact [11] and that $x_i^2 \leqslant 1$ μ-a.e. for any limit probability μ. In the second step we use a GKS inequality to show that, in fact, $x_i = \pm 1$ μ-a.e. Finally, we prove that μ, which is obviously translation invariant, satisfies (3).

To prove (4) we first establish

$$\int \exp(\lambda x_i^2)\, d\mu^\nu \leqslant \lim_{L \uparrow \mathbf{Z}^d} \left[\int \exp\left(\lambda \sum_{i \in L} x_i^2\right) d\mu_{L,P}^\nu \right]^{1/|L|}. \tag{5}$$

Let L be the d-dimensional torus of circumference 2^n, and let A_k, $k = \phi, 0, 1, \ldots, d-1$, be the operator on $L^2(\mathbf{R}^{2^n(d-1)}, d^{2^n(d-1)}x)$ with kernel

$$A_K(x, y) = \int \exp\left(-\beta \sum_{i \in L'} (x_i - z_i)^2 - \beta \sum_{i \in L'} (z_i - y_i)^2 + a_K \right) d\mu_{L',P}^\nu(z),$$

where L' is the $d-1$-dimensional torus with circumference 2^n, and $a_\phi = 0$,

$$a_0 = \lambda z_{(1, \ldots, 1)}^2, \quad a_K = \lambda \sum_{i_1, \ldots, i_K = 1}^{2^n} z_{(i_1, \ldots, i_K, 1, \ldots, 1)}^2.$$

Using

$$\int \exp\left(-\beta(z_i - y)^2\right) \exp\left(-\beta(y - z_{i+1})^2\right) dy = c \exp\left(-\beta/2 (z_i - z_{i+1})^2\right),$$

we see that we may write

$$\int \exp(\lambda x_i^2)\, d\mu_{L,P}^\nu = \frac{\operatorname{Tr}\left(A_\phi^{2^n - 1} A_0\right)}{\operatorname{Tr}\left(A_\phi^{2^n}\right)}.$$

Then, by repeated use of the cyclicity of traces, the invariance of our measure under lattice translation and rotations, and the Schwarz inequality for traces we find

$$\frac{\operatorname{Tr}\left(A_\phi^{2^n-1}A_0\right)}{\operatorname{Tr}\left(A_\phi^{2^n}\right)} = \frac{\operatorname{Tr}\left(A_\phi^{2^{n-1}}A_0A_\phi^{2^{n-1}-1}\right)}{\operatorname{Tr}\left(A_\phi^{2^n}\right)} \leqslant \left[\frac{\operatorname{Tr}\left(A_\phi^{2^{n-1}-1}A_0^2A_\phi^{2^{n-1}-1}\right)}{\operatorname{Tr}\left(A_\phi^{2^n}\right)}\right]^{1/2}$$

$$= \left[\frac{\operatorname{Tr}\left(A_\phi^{2^{n-1}}A_0^2A_\phi^{2^{n-1}-2}\right)}{\operatorname{Tr}\left(A_\phi^{2^n}\right)}\right]^{1/2} \leqslant \cdots \leqslant \left[\frac{\operatorname{Tr}\left(A_0^{2^n}\right)}{\operatorname{Tr}\left(A_\phi^{2^n}\right)}\right]^{1/2^n}$$

$$= \left[\frac{\operatorname{Tr}\left(A_\phi^{2^{n-1}}A_1A_\phi^{2^{n-1}-1}\right)}{\operatorname{Tr}\left(A_\phi^{2^n}\right)}\right]^{1/2^n} \leqslant \cdots \leqslant \left[\frac{\operatorname{Tr}\left(A_1^{2^n}\right)}{\operatorname{Tr}\left(A_\phi^{2^n}\right)}\right]^{1/4^n}$$

$$\leqslant \cdots \leqslant \left[\frac{\operatorname{Tr}\left(A_{d-1}^{2^n}\right)}{\operatorname{Tr}\left(A_\phi^{2^n}\right)}\right]^{1/|L|} = \left[\int \exp\left(\lambda \sum_{i \in L} x_i^2\right) d\mu_{L,P}^\nu\right]^{1/|L|}.$$

The GKS inequalities [12, p. 289] imply that $\int \exp(\lambda x_i^2)\, d\mu_L^\nu \leqslant \int \exp(\lambda x_i^2)\, d\mu_{L,P}^\nu$ if $\lambda \geqslant 0$. (5) now follows on letting $L \uparrow \mathbf{Z}^d$.

Then we note that

$$\left[\int \exp\left(\lambda \sum_{i \in L} x_i^2\right) d\mu_{L,P}^\nu\right]^{1/|L|}$$

$$= \left[\frac{\int \exp\left(-\mathcal{H}_{L,P}(x_1, \ldots, x_{|L|}) - \sum (x_i^2 - 1)^2/\nu + \lambda \sum x_i^2\right) dx_1, \ldots, dx_{|L|}}{\int \exp\left(-\mathcal{H}_{L,P}(x_1, \ldots, x_{|L|}) - \sum (x_i^2 - 1)^2/\nu\right) dx_1 \ldots dx_{|L|}}\right]^{1/|L|}$$

$$\leqslant \frac{\int \exp\left(hx + (\lambda - m_0^2)x^2 - (x^2 - 1)^2/\nu\right) dx}{\int \exp\left(hx - (2d\beta + m_0^2)x^2 - (x^2 - 1)^2/\nu\right) dx},$$

where we have used $\exp(-(x_i - x_j)^2) \leqslant 1$ to decouple the integrand in the numerator, and $(x_i - x_j)^2 \leqslant 2x_i^2 + 2x_j^2$ for the denominator. Our last inequality together with (5) implies (4), since $\exp(-(x^2 - 1)^2/\nu)\, dx/N \to \frac{1}{2}(\delta(x + 1) + \delta(x - 1))$ as $\nu \to 0$.

As we noted, (4) implies $x_i^2 \leqslant 1$ μ-a.e. for any limit probability μ. However, the GKS inequalities imply [12, p. 286]

$$\int x_i^2\, d\mu = \lim_{\nu_k \to 0} \int x_i^2\, d\mu^{\nu_k} \geqslant \lim_{\nu_k \to 0} \int x_i^2\, d\mu_L^{\nu_k} = 1, \qquad (L \ni i).$$

since $\exp(-(x^2 - 1)^2/\nu)dx/N \to \frac{1}{2}(\delta(x + 1) + \delta(x - 1))$ as $\nu \to 0$ implies $\mu_L^\nu(\cdot)$ converges to $P_L(\cdot)$. Since μ is a probability measure, $\int x_i^2\, d\mu \geqslant 1$ and $x_i^2 \leqslant 1$ are compatible only if $x_i = \pm 1$ μ-a.e.

To see that the limit probability μ on $\{-1, 1\}^{\mathbf{Z}^d}$ satisfies (3), let us define π_K for $K \subseteq \mathbf{Z}^d$ to be the operator on $\mathbf{R}^{\mathbf{Z}^d}$ which is defined coordinatewise and takes $x_i \to -x_i$ if $i \in K$, and $x_j \to x_j$ if $j \notin K$. It is easy to check that

$$d\mu^\nu(\pi_K x) = \exp(-\mathcal{H}_L(\pi_K x) + \mathcal{H}_L(x))\, d\mu^\nu(x)$$

for any $L \supseteq K \cup \partial K$. We note that $\exp(-\mathcal{H}_L(\pi_K x) + \mathcal{H}_L(x))$ is well defined, since it depends only on those coordinates of x which are indexed by $i \in K \cup \partial K$. Furthermore, it is independent of ν, hence

$$d\mu(\pi_K x) = \exp(-\mathcal{H}_L(\pi_K x) + \mathcal{H}_L(x))\, d\mu(x). \tag{6}$$

This will imply (3) once we verify that $\mu(B) > 0$ for any finite configuration B, which will allow us to form $\mu(\cdot\,|B)$. Let A be the configuration on the finite set L which assigns $+1$ to all σ_i, $i \in L$. By GKS [12, p. 286],

$$\mu(A) = \int \prod_{i \in L} (1 + x_i)/2^{|L|}\, d\mu = \lim_{\nu_k \to 0} \int \prod_{i \in L} (1 + x_i)/2^{|L|}\, d\mu^{\nu_k}$$

$$\geqslant \lim_{\nu_k \to 0} \int \prod_{i \in L} (1 + x_i)/2^{|L|}\, d\mu_L^{\nu_k} = P_L(A) > 0,$$

which is positive by inspection. That $\mu(B) > 0$ for any configuration B on L now follows from (6).

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