AN UNCOUNTABLE NOETHERIAN RING WITH FREE ADDITIVE GROUP

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ABSTRACT. An example is given of an uncountable Noetherian ring whose additive group is free. This answers a question posed indirectly by L. Fuchs.

In [2] on page 311 we read: "we do not know of any uncountable Noetherian ring whose additive group is free." Our purpose is to give an example of such a ring.

Let A be the commutative associative polynomial ring $Z[x_1, \dots, x_i, \dots]$, $i \in I$, where Z is the ring of integers, each x_i is indeterminate, and I is the set of positive ordinals less than ω_1 , the first uncountable ordinal. Let S be the set consisting of those members of A which are relatively prime to each positive integer. Let R be the subring of the field $Z(x_1, \dots, x_i, \dots)$ generated by those members which can be written in the form p/q with $p \in A$ and $q \in S$. We will prove the following theorem.

THEOREM 1. *The ring R is an uncountable Noetherian ring with free additive group.*

In order to prove this theorem, we shall need the following theorem which was proved in a more general form by Paul Hill (see Theorem 3 of [3]).

THEOREM A. *Let ω_1 be the first uncountable ordinal. If $F_1 \subseteq F_2 \subseteq \dots \subseteq F_j \subseteq \dots$, $j < \omega_1$, is an ascending chain of free subgroups of an abelian group G , indexed by the positive ordinals less than ω_1 , such that*

- (i) $F_k = \bigcup_{j < k} F_j$ if k is a limit ordinal less than ω_1 ,
- (ii) $G = \bigcup_{j < \omega_1} F_j$,
- (iii) $|F_j| < \aleph_1$ for each j , then G is free if F_{j+1}/F_j is free for each positive $j < \omega_1$.

In this paragraph we explain some terminology. Whenever an element in R is expressed in rational form p/q , it is assumed that p is in A and q is in S . Let $x_0 = 1$, the identity element in R . For $j \in I$, let $(F_j, +)$ be the subgroup of $(R, +)$ generated by those members which can be written in the form p/q where p and q are polynomials in $\{x_i: 0 \leq i < j\}$. If $j \in I$ and $p \in F_{j+1} \cap A$, by $\deg_j p$ we mean the degree of p considered as a polynomial in x_j over

Received by the editors March 9, 1977.

AMS (MOS) subject classifications (1970). Primary 20K20, 16A46.

Key words and phrases. Free group, torsion-free group, pure independent set.

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$F_j \cap A$. Finally, a set is a *pure independent* set in a group G if its members are linearly independent and generate a pure subgroup of G .

Before proceeding to the proof of Theorem 1, we establish two lemmas.

LEMMA 1. *If $j \in I$ and L is a finite subset of F_{j+1} , then L is linearly dependent, modulo F_j , on a finite set $K = \{f_k/g\} \subseteq F_{j+1}$ such that each f_k is a finite product of x_i 's, $\deg_j f_k \neq \deg_j g$ for each k , and the set $\{f_k\}$ is a pure independent set in $(A, +)$.*

PROOF. We may write $L = \{p_t/q\}$ where each p_t and q are polynomials in $x_0, \dots, x_j$. Suppose $\deg_j q = 0$. We may then suppose that $\deg_j p_t$ is positive for each t . The set $\{p_t\}$ is linearly dependent on a pure independent finite set $\{f_k\}$ where each f_k is a finite product of x_i 's, $i \leq j$, and no f_k is in F_j . If $g = q$, the set $K = \{f_k/g\}$ satisfies the lemma. Suppose, then, $\deg_j q = r > 0$, and write $q = a + mbx_j^r$ where a is a polynomial in $F_{j+1} \cap A$ with $\deg_j a < r$, m is a nonzero integer, and b is a polynomial in $S \cap F_j$. We may assume that each p_t is a single finite product of x_i 's (since each p_t is a linear combination of such products). Now we form a new set $L' = \{p'_t/qb\}$ as follows: if $\deg_j p_t \neq \deg_j q$, let $p'_t = p_t b$; if $p_t = c_t x_j^r$ for some nonzero c_t in $A \cap F_j$, let $p'_t = -c_t a$. In the latter case $p'_t/qb = -c_t a/qb = mp_t/q - c_t/b$ and c_t/b is in F_j . Hence, L is linearly dependent on L' , modulo F_j , and no p'_t , when expressed as a polynomial in x_j over $A \cap F_j$, has a term of degree r , the degree of qb . Let $\{f_k\}$ be the minimal set of finite products of x_i 's which generate (over Z) the set $\{p'_t\}$. If $g = qb$, the set $\{f_k/g\}$ satisfies the lemma.

LEMMA 2. *Let K be as in Lemma 1 and let $\bar{K}$ be its image in F_{j+1}/F_j . If E is the set of elements in F_{j+1}/F_j linearly dependent on $\bar{K}$, then E is a free subgroup.*

PROOF. First, F_{j+1}/F_j is torsion-free. Suppose $p/q \in F_{j+1}$ and $np/q = p_1/q_1 \in F_j$ for some nonzero integer n . Then $npq_1 = p_1q$ and $(n, q) = 1$ implies that n divides p_1 in A . Thus, n can be cancelled, $p/q \in F_j$ and F_{j+1}/F_j is torsion-free, as desired. Next, since $\bar{K}$ is finite, it generates a free subgroup E_1 . We will show that E/E_1 is bounded and it will follow that E is free (see Proposition 18.3 of [1]). Suppose the coset $p/q + F_j$ is in E with $p/q \in F_{j+1}$. From the description of K , $\bar{K}$, and E given in the lemmas, we have $np/q = \sum n_k f_k/g + p_1/q_1$ for some $p_1/q_1 \in F_j$ and integers n_k, n with $n \neq 0$. It follows that n divides $q_1(\sum n_k f_k) + gp_1$ in A . If $\deg_j g = 0$, then $\deg_j f_k > 0$ for each k and n divides $\sum n_k f_k$ in A since $(n, q_1) = 1$. Since $\{f_k\}$ is a pure independent set in A , n divides each n_k and $p/q + F_j$ is in E_1 . However, if $\deg_j g = r > 0$, write $g = a + mbx_j^r$ where $a \in F_{j+1} \cap A$ with $\deg_j a < r$, m is a nonzero integer, and $b \in F_j \cap S$. Since $\deg_j f_k \neq r$ for each k , n must divide $mbx_j^r p_1$ in A . Since $(b, n) = 1 = (p_1, n)$, n divides m in Z . But m has only a finite number of integral divisors. Therefore, E/E_1 is bounded and E is free, as desired.

PROOF OF THEOREM 1. Since I is uncountable, so is the ring R . It is easy to

see that every ideal of R is generated by an integer. As a result, R is Noetherian. We now show that the additive group of R is free. We employ Theorem A. Let G be the additive group of R and let F_j , $1 \leq j < \omega_1$, be as defined in the paragraph after the statement of Theorem A. Clearly, conditions (i), (ii) and (iii) of Theorem A are satisfied. We must show that F_j and F_{j+1}/F_j are free for each j in I . Let j be a fixed ordinal in I . Now F_{j+1}/F_j is torsion-free, as was shown at the beginning of the proof of Lemma 2. Since this group is countable, by Pontryagin's Theorem (see Proposition 19.1 of [1]), it will be free if every subgroup of it of finite rank is free. Let H be such a subgroup. Let L be a finite subset of F_{j+1} which maps, by the natural map, to a maximal linearly independent set $\bar{L}$ in H . Relative to this L , let K , $\bar{K}$ and E be as in Lemmas 1 and 2. Since L is linearly dependent on K , modulo F_j by Lemma 1, $\bar{L}$ is linearly dependent on $\bar{K}$ in F_{j+1}/F_j and $E \supseteq H$. By Lemma 2, E is free. Since a subgroup of a free group is free, H is free. Therefore, F_{j+1}/F_j is free for each j in I . Finally, we show that F_j is free for each j in I . We distinguish three cases: $j = \omega$, $j < \omega$ and $j > \omega$ where ω is the first infinite ordinal. For positive ordinal j less than ω , F_{j+1}/F_j is free and $F_{j+1} = F_j \oplus G_j$ for some free subgroup G_j . Set $F_1 = G_0$. This group is isomorphic to the additive group of $\mathbb{Z}$ and hence free. Thus, $F_\omega = \bigoplus G_j$, $0 \leq j < \omega$, and F_ω is free. Since a subgroup of a free group is free, F_j is free for $j < \omega$. Suppose $\omega < j < \omega_1$. Since j is countable, the set $\{i: 1 \leq i < j\}$ may be placed in one-to-one correspondence with the natural numbers. From their definitions it follows that F_j is isomorphic to F_ω . Therefore, F_j is free for $\omega < j < \omega_1$. The proof is complete.

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