A NOTE ON SYLOW INTERSECTIONS

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ABSTRACT. Let p be a prime number, G a finite group whose order is divisible by p , and S a Sylow p -subgroup of G . We say that G satisfies $(*)$ if there exists $g \in G$ such that $S \cap S^g = O_p(G)$. There are examples of primes p and finite groups G that do not satisfy $(*)$. In this note we discuss several related properties satisfied in general, as well as giving sufficient conditions for G to satisfy $(*)$.

1. Introduction. Let G be a finite group whose order is divisible by a prime number p , and let S be a Sylow p -subgroup of G . Laffey [5] proved that if $O_p(G) = 1$ and $R \in \text{Syl}_p(G)$ is such that $|R \cap S|$ is minimal, then $Z(R) \cap Z(S) = 1$. Generalizing his method we get

PROPOSITION 1. Assume $O_p(G) = 1$, and let P be a strongly closed subgroup of S with respect to G . Choose $g \in G$ to minimize $|\Omega(P \cap P^g)|$. Then $C_p(\Omega(P)) \cap C_{p^*}(\Omega(P^g)) = 1$. If, in addition, $\Omega(P)$ is abelian, then $P \cap S^g = 1$.

Recall [3] that an Ω -functor (for p) on G is a mapping W of the set of p -subgroups of G into itself that satisfies the following conditions: for every p -subgroup P of G :

- (i) $W(P) \subseteq P$;
- (ii) $W(P) \neq 1$ if $P \neq 1$;
- (iii) $W(P^g) = (W(P))^g$ for every $g \in G$; and
- (iv) $W(P_1) \subseteq W(P)$ for every p -subgroup P_1 of G such that $W(P_1) \subseteq P$.

Herzog [3] introduced the concept of Ω -functor and used it to prove that if $O_p(G) = 1$, and $\Omega(S)$ is abelian, then there exists $R \in \text{Syl}_p(G)$ such that $R \cap S = 1$. Laffey [6] used this concept to prove a generalization of the above quoted result of Herzog [3] and his own result in [5]. We discuss Ω -functors still further, give some more examples, and prove the following result (compare with Laffey [6]).

PROPOSITION 2. Assume $O_p(G) = 1$. Choose $R \in \text{Syl}_p(G)$ to minimize $|\Omega(R \cap S)|$. Then for every Ω -functor W , $O_p(H) = 1$, where $H \equiv \langle W(R), W(S) \rangle$.

We say that G satisfies $(*)$ if there exists $g \in G$ such that $S \cap S^g = O_p(G)$.

Received by the editors November 1, 1976.

AMS (MOS) subject classifications (1970). Primary 20D20.

¹This note is part of the author's Ph.D. thesis, done at the University of Tel-Aviv under the supervision of Professor M. Herzog.

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Trying to find sufficient conditions for G to satisfy $(*)$, we were led by Herzog [2] to define the following set of p -subgroups of G . Let Π^* denote the set of nonidentity p -subgroups P of G , such that (i) $N_G(P)$ contains a Sylow p -subgroup of G , (ii) $P = O_p(N_G(P))$, and (iii) $N_G(P)$ satisfies $(*)$. Note that given any nonidentity p -subgroup Q , normal in some Sylow p -subgroup of G , then $P = O_p(N_G(Q))$ satisfies both (i) and (ii). Clearly, if G is a counter-example to $(*)$ of minimal order, then any nonidentity p -subgroup P of G satisfies (iii).

It is the following property which reveals the relevance of the set Π^* to our problem.

LEMMA 3. *Let $P \in \Pi^*$, and Q be a p -subgroup of G . Then $P \cap Q = R \cap Q$ for some $R \in \text{Syl}_p(G)$.*

This result immediately yields

COROLLARY 4. *Assume $O_p(G) = 1$, and let $P \in \Pi^*$.*

- (i) *If G does not satisfy $(*)$, then $P \cap Q \neq 1$ for every $Q \in \text{Syl}_p(G)$.*
- (ii) *If $p\text{-rank}(P) = 1$, then G satisfies $(*)$.*
- (iii) *If P is strongly closed in S with respect to G , and $\Omega(P)$ is abelian, then G satisfies $(*)$.*

2. Proof of Proposition 1 and related results. Let χ denote a family of subsets of G such that if $A \in \chi$ and $g \in G$, then $A^g (\equiv \{a^g | a \in A\})$ belongs to χ . We call such a family a normal family for G . For a subgroup H of G define $\chi(H) \equiv \{A \in \chi | A \subseteq H\}$, and set $X(H) \equiv \langle \chi(H) \rangle$. Clearly, for every $g \in G$, $\chi(H^g) = (\chi(H))^g$. Also, $\chi(H \cap K) = \chi(H) \cap \chi(K)$.

PROPOSITION 5. *Let R be a Sylow p -subgroup of G such that $X(R \cap S) = X(T \cap U)$ for every $T, U \in \text{Syl}_p(G)$ satisfying $\chi(R \cap S) \supseteq \chi(T \cap U)$. If K is a p -subgroup of G normalized by $\langle X(R), X(S) \rangle$, then $X(K) \subseteq O_p(G)$.*

PROOF. Let R', S' be Sylow p -subgroups of $H \equiv \langle X(R), X(S) \rangle K$ containing $X(R), X(S)$, respectively, and choose $h \in H$ such that $(R')^h = S'$. Let T be a Sylow p -subgroup of G , and T' a Sylow p -subgroup of H containing $T \cap H$. Choose $k \in H$ such that $(R')^k = T'$, and U a Sylow p -subgroup of G containing $U' \equiv (T')^k$.

As $\chi(S) \subseteq S'$, $|\chi(U)| = |\chi(S)| = |\chi(S')| = |\chi(U')|$ yields $\chi(U) = \chi(U')$. Thus,

$$\begin{aligned} \chi(U \cap T) &= \chi(U) \cap \chi(T) = \chi(U') \cap \chi(T) = \chi(U' \cap H) \cap \chi(T) \\ &= \chi(U') \cap \chi(H \cap T) \subseteq \chi(U') \cap \chi(T') = \chi(U' \cap T') \\ &= (\chi(S' \cap R'))^k = (\chi(S \cap R))^k, \end{aligned}$$

so, by our assumption, $X(U \cap T) = X(U' \cap T')$, whence

$$X(K) \subseteq X(O_p(H)) \subseteq X(U' \cap T') \subseteq T.$$

As T is an arbitrary Sylow p -subgroup of G , $X(K) \subseteq O_p(G)$.

LEMMA 6. Choose $g \in G$ such that $S^g = R$ satisfies $X(R \cap S) = X(T \cap U)$ for every $T, U \in \text{Syl}_p(G)$ such that $\chi(R \cap S) \supseteq \chi(T \cap U)$. Then $X(C_R(X(R)) \cap C_S(X(S))) \subseteq O_p(G)$. Moreover, $X(P \cap P^g) \subseteq O_p(G)$ for every $P \subseteq S$ such that $X(P)$ centralizes $X(S)$.

PROOF. As $K \equiv C_R(X(R)) \cap C_S(X(S))$ is a p -subgroup of G normalized by $\langle X(R), X(S) \rangle$, the first assertion follows from Proposition 5. Now, our assumption yields $\chi(P \cap P^g) \subseteq \chi(K)$, whence the second assertion follows.

Denote by χ_∞ the family of all one element subsets of G , and by χ_0 the family of all one element subsets of G , consisting of an element of prime order. Lemma 6 with $\chi = \chi_0$ or $\chi = \chi_\infty$ yields

COROLLARY 7. Assume $O_p(G) = 1$.

(1) Choose $R \in \text{Syl}_p(G)$ to minimize $|\Omega(R \cap S)|$. Then

(i) $C_R(\Omega(R)) \cap C_S(\Omega(S)) = 1$,

(ii) $Z(R) \cap Z(S) = 1$, and

(iii) (Herzog) if $\Omega(S)$ is abelian, then $R \cap S = 1$.

(2) (Laffey) Choose $R \in \text{Syl}_p(G)$ to minimize $|R \cap S|$. Then $Z(R) \cap Z(S) = 1$.

Taking for χ the set of one element subsets of G , consisting of an element of order p , conjugate to an element of P , yields Proposition 1 as a corollary of Lemma 6.

Let T be an elementary abelian subgroup of S . If $p = 2$ suppose that T is of maximal order, whereas if $p \neq 2$ suppose that T is of maximal order subject to being normal in S . Define the normal family $\chi \equiv \{\{t^g\} | t \in T, g \in G\}$, and set $A \equiv C_S(X(S))$.

COROLLARY 8. If $O_p(G) = 1$, then there exists $g \in G$ such that $A \cap A^g = 1$.

PROOF. We quote Lemma 6 to get $X(A \cap A^g) = 1$. Hence the corollary follows once we show that $X(A) = \Omega(A)$. For $p = 2$ this is clear, whereas for $p \neq 2$ we quote Lemma 5.4.14 of [1].

3. Ω -functors. A definition of Ω -functor appears in the introduction. Obvious examples are the Ω -functors Ω_i defined by $\Omega_i(P) \equiv \langle x \in P | x^{p^i} = 1 \rangle$. Similar (more complicated) examples are discussed in [6]. If χ is any normal family for G containing χ_0 then $W_\chi(P) = X(P)$ defines an Ω -functor in G . One can define similar examples, in analogy with those discussed in [6], as follows: Let n be a positive integer, and $\chi_i, i = 1, \dots, n$, normal families in G satisfying $\chi_0 \subseteq \chi_1 \subseteq \dots \subseteq \chi_n$. For every p -subgroup P of G define $W(P)$ as follows: If there exist integers $1 \leq i < n$ such that $|X_i(P)| < |X_i(S)|$ let k be the minimal such integer and define $W(P) = W_{\chi_k}(P)$. If there exists no such integer define $W(P) = W_{\chi_n}(P)$.

We note that for every Ω -functor W and every p -subgroup P of G , $W(P)$ is weakly closed in P with respect to G , and that $W(P) \supseteq \Omega(P)$. Proposition 2 is now an immediate consequence of Proposition 5, for, taking $\chi = \chi_0$, it is clear

that $K \equiv O_p(H)$ is a p -subgroup of G normalized by $H = \langle W(R), W(S) \rangle \supseteq \langle \Omega(R), \Omega(S) \rangle = \langle X(R), X(S) \rangle$, whence $\Omega(K) = X(K) \subseteq O_p(G) = 1$, and we are done.

4. Proofs and further results. In the introduction we defined the property $(*)$ and the set Π^* and stated Lemma 3. The proof of this lemma is straightforward. Indeed, let T be a Sylow p -subgroup of $N_G(P)$ containing $N_G(P) \cap Q$. As $N_G(P)$ satisfies $(*)$, we can choose $R \in \text{Syl}_p(N_G(P))$ such that $R \cap T = O_p(N_G(P)) = P$. Now, $R \in \text{Syl}_p(G)$ and

$$P \cap Q \subseteq R \cap Q = R \cap N_G(P) \cap Q \subseteq R \cap T \cap Q = P \cap Q,$$

so we are done.

Another relevant set of p -subgroups of G is the set Π' of all subgroups P of G such that $P = \cap \Sigma$ for some $\Sigma \subseteq \text{Syl}_p(G)$. For $P \in \Pi'$ we define $k(P)$ to be the minimal number k such that there exists $\Sigma' \subseteq \text{Syl}_p(G)$ with $\cap \Sigma' = P$, and $|\Sigma'| = k$ (see [2]). It is clear that $\Pi^* \subseteq \Pi'$, and that $P \in \Pi'$ with $k(P) = 1$ (namely, a Sylow p -subgroup of G) belongs to Π^* .

LEMMA 9. *Assume that every $P \in \Pi'$ with $k(P) = 2$ belongs to Π^* . Then $k(P) \leq 2$ for every $P \in \Pi'$, and $\Pi^* = \Pi'$.*

PROOF. Let U be a subgroup in Π' with $k(U) > 1$, such that if $P \in \Pi'$ and $k(P) < k(U)$ then $P \in \Pi^*$. Choose then $\Sigma \subseteq \text{Syl}_p(G)$ such that $\cap \Sigma = U$ and $|\Sigma| = k(U)$, take $Q \in \Sigma$ and set $P = \cap (\Sigma \setminus \{Q\})$. Clearly $P \in \Pi^*$, so by Lemma 3, $U = P \cap Q = R \cap Q$ for some $R \in \text{Syl}_p(G)$, proving $k(U) = 2$. The conclusions of the lemma follow now by a suitable induction argument.

COROLLARY 10 (HERZOG). *If for every $R, T \in \text{Syl}_p(G)$, $R \cap T \trianglelefteq R$, then $k(P) \leq 2$ for every $P \in \Pi'$.*

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