THE AUTOMORPHISM GROUP OF ${}^2F_4(2)'$

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ABSTRACT. We give a new proof that $\text{Aut}({}^2F_4(2)') \cong \text{Aut}({}^2F_4(2)) \cong {}^2F_4(2)$ (see [4]).

In [4] Griess and Lyons showed that $\text{Aut}({}^2F_4(2)') \cong \text{Aut}({}^2F_4(2)) \cong {}^2F_4(2)$. In this note we would like to give a more transparent proof of this fact based on the relationship of ${}^2F_4(2)$ to the corresponding generalized 8-gon.

We begin by presenting the following terminology: Let $\mathcal{G}$ be an undirected graph with vertex set $V(\mathcal{G})$ and G a subgroup of $\text{Aut}(\mathcal{G})$. We denote by $\mathcal{G}(v)$ the set of neighbors of a vertex $v \in V(\mathcal{G})$, by $G(v)$ the stabilizer of v in G and by $G(v)^{\mathcal{G}(v)}$ the permutation group induced by $G(v)$ on $\mathcal{G}(v)$. For each vertex $v \in V(\mathcal{G})$ and each $i \in \mathbb{N}$, let $G_i(v) = \{g \in G \mid g \in G(u) \text{ if } \partial(u, v) \leq i\}$, where $\partial(u, v)$ denotes the distance between u and v . A sequence $(v_0, \dots, v_s)$ of $s+1$ vertices v_i is called an s -path if $v_i \in \mathcal{G}(v_{i-1})$ for $1 \leq i \leq s$ and $v_i \neq v_{i-2}$ for $2 \leq i \leq s$. For each vertex $v \in V(\mathcal{G})$ and each $s \in \mathbb{N}$, let $W_s(v)$ be the set of s -paths $(v_0, \dots, v_s)$ with $v = v_0$. We say (as in [1]) that $\mathcal{G}$ is locally (G, s) -regular if for every vertex $v \in V(\mathcal{G})$, $G(v)$ acts regularly on $W_s(v)$.

Now let $\mathcal{K}$ be the incidence graph of the generalized 8-gon associated with ${}^2F_4(2)$. The vertex set $V(\mathcal{K})$ of $\mathcal{K}$ consists of all maximal subgroups of ${}^2F_4(2)$ containing a 2-Sylow group of ${}^2F_4(2)$; two vertices are joined by an edge if and only if their intersection is a 2-Sylow group of ${}^2F_4(2)$. The group ${}^2F_4(2)$ acts faithfully on $V(\mathcal{K})$, so we may consider ${}^2F_4(2)$ to be a subgroup of $\text{Aut}(\mathcal{K})$. Let $H = {}^2F_4(2)$, $J = {}^2F_4(2)'$ and $K = \text{Aut}(\mathcal{K})$. It is well known that $\mathcal{K}$ is connected and locally $(H, 9)$ -regular (see, for instance, [2, (15.5.1)] and [5, (4.2)]). We have

$$\text{Syl}_2(J) = \{J(u, v) \mid u, v \in V(\mathcal{K}), u \in \mathcal{K}(v)\},$$

where $J(u, v) = J(u) \cap J(v)$. Let (u, v, w) be an arbitrary 2-path in $\mathcal{K}$. Since $J(u, v)$ acts transitively on $\mathcal{K}(v) - \{u\}$, $[J(u, v): J(u, v, w)] = |\mathcal{K}(v)| - 1 = 2$ or 4 , where $[J(u, v): J(u, v, w)]$ denotes the index of $J(u, v, w) = J(u, v) \cap J(w)$ in $J(u, v)$. Let $\{x_0, x_1\}$ and $\{y_0, y_1\}$ be arbitrary edges of $\mathcal{K}$. We may assume that $\partial(x_0, y_0) \leq \partial(x_i, y_j)$ for all i and $j = 0$ or 1 ; let $t = \partial(x_0, y_0)$. Then there exists a unique $(t+2)$ -path $(v_0, v_1, \dots, v_{t+1}, v_{t+2})$

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with $(v_0, v_1) = (x_1, x_0)$ and $(v_{t+1}, v_{t+2}) = (y_0, y_1)$. We have $J(x_0, x_1) \cap J(y_0, y_1) \leq J(v_0, \dots, v_{t+2})$. If $t \geq 1$, then $[J(x_0, x_1) \cap J(y_0, y_1)] \geq 8$ since $J(v_0, v_1, v_2)$ acts transitively on $\mathcal{H}(v_2) - \{v_1\}$. Thus two edges $\{u, v\}$ and $\{x, y\}$ of $\mathcal{H}$ have a vertex in common if and only if $[J(u, v) \cap J(x, y)] \leq 4$. It follows that an arbitrary automorphism of J induces a permutation on the edge set of $\mathcal{H}$ mapping incident edges to incident edges. By [6, Theorem 1, Case B], such a permutation is induced by an automorphism of $\mathcal{H}$. If L is the normal subgroup of $\text{Aut}(J)$ acting trivially on $\text{Syl}_2(J)$, then $L \cap \text{Inn}(J) = 1$ and hence $L = 1$. Similarly, $\text{Aut}(H)$ acts faithfully on $\text{Syl}_2(H)$. Thus, in order to show that $\text{Aut}(J) \cong \text{Aut}(H) \cong H$, it suffices to show that $H = K$.

Let $(v_1, \dots, v_8)$ be an arbitrary 7-path in $\mathcal{H}$. We claim that $K_1(v_1, \dots, v_8) = 1$, where

$$K_1(v_1, \dots, v_8) = K_1(v_1) \cap \dots \cap K_1(v_8).$$

Since $\mathcal{H}$ is connected, it is enough to show that $K_1(v_1, \dots, v_8) \leq K_1(v_2, \dots, v_8, x)$ for every vertex $x \in \mathcal{H}(v_8) - \{v_7\}$. Pick an arbitrary vertex x in $\mathcal{H}(v_8) - \{v_7\}$. Let $\mathcal{H}(v_1) = \{v_2, y_1, \dots, y_m\}$ and $\mathcal{H}(x) = \{v_8, u_1, \dots, u_m\}$, where $m = |\mathcal{H}(v_1)| - 1 = 2$ or 4 . We can order the vertices $y_1, \dots, y_m$ so that $\partial(y_i, u_i) = 6$ for $1 \leq i \leq m$. Since the girth of $\mathcal{H}$ (i.e., the length of the shortest closed path in $\mathcal{H}$) is 16, it follows that $K(y_i) \cap K(x) \leq K(u_i)$ for $1 \leq i \leq m$ and thus $K_1(v_1, \dots, v_8) \leq K_1(x)$.

Since $H_1(v_1, \dots, v_7)$ acts transitively on $\mathcal{H}(v_8) - \{v_7\}$ and $K_1(v_1, \dots, v_8) = 1$, it follows that $K_1(v_1, \dots, v_7)$ is isomorphic to a transitive subgroup of the symmetric group S_k where $k = |\mathcal{H}(v_8)| - 1 = 6 - m = 2$ or 4 . Thus

$$O_2(K_1(v_1, \dots, v_7)) \not\leq K_1(v_8).$$

Since

$$\begin{aligned} K_1(v_1, \dots, v_7) &\triangleleft K_1(v_2, \dots, v_7) \triangleleft \dots \triangleleft K_1(v_6, v_7) \\ &\triangleleft K_1(v_7) \triangleleft K(v_7, v_8), \end{aligned}$$

we have $O_2(K(v_7, v_8)) \not\leq K_1(v_8)$.

Now let (u, v) be an arbitrary 1-path in $\mathcal{H}$, $A = K(u, v)$ and $B = O^2(K_1(u, v))$. Let x be either u or v . Since $O_2(A) \not\leq K_1(x)$, we can pick a vertex $y \in \mathcal{H}(x)$ and an element $a \in O_2(A)$ such that $a \notin K(y)$. Let $C = O^2(K_1(x, y))$. We have $C^a = a^{-1}Ca \triangleleft K_1(x)$. Since CC^a/C^a is a subgroup of the 2-group $(CO_2(A) \cap K_1(x))/C^a$, $C/C \cap C^a$ is also a 2-group. Since $O^2(C) = C$, it follows that $C = C^a$ and thus $C \triangleleft \langle K(x, y), K(x, (y)a) \rangle$. Since $K(x)^{\mathcal{H}(x)}$ is 2-transitive,

$$\langle K(x, y), K(x, (y)a) \rangle = K(x).$$

If $b \in K(x)$ is picked such that $\{(x)b, (y)b\} = \{u, v\}$, then $C = C^b = B$. Thus $B \triangleleft K(x)$ for $x = u$ and v . Since $\mathcal{H}$ is connected, $\langle K(u), K(v) \rangle$ acts transitively on the edge set of $\mathcal{H}$. Thus $B = 1$ and $K_1(u, v)$ is a 2-group (compare [3, (2.3)]).

Finally, let (u, v, w) be an arbitrary 2-path in $\mathcal{H}$ beginning with a vertex u of valency five. The group $H_1(u, v)$ acts cyclicly on $\mathcal{H}(w) - \{v\}$. Since $K_1(u, v)$ is a 2-group and $K_1(u, v) \triangleleft K_1(v) \triangleleft K(v, w)$, we have

$$H_1(u, v) \leq K_1(u, v) \leq O_2(K(v, w)).$$

Thus $O_2(K(v, w)^{\mathcal{H}(w)})$ contains a cycle of order four. It follows that $K(w)^{\mathcal{H}(w)}$ is isomorphic to the group of affine transformations of the field with five elements, i.e., $K(w)^{\mathcal{H}(w)} = H(w)^{\mathcal{H}(w)}$. Thus

$$K(v_0, \dots, v_9) = K_1(v_1, \dots, v_8)$$

for every 9-path $(v_0, \dots, v_9)$ in $\mathcal{H}$. It follows that $\mathcal{H}$ is locally $(K, 9)$ -regular and therefore $|K(v)| = |W_s(v)| = |H(v)|$ for every vertex $v \in V(\mathcal{H})$. Hence $|K| = |H|$.

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