CONVERGENCE OF CERTAIN COSINE SUMS IN THE METRIC SPACE L

BABU RAM

ABSTRACT. We consider here the L^1 convergence of Rees-Stanojević cosine sums to a cosine trigonometric series belonging to the class S defined by Sidon and deduce as corollaries some previously known results from our result.

1. **Introduction.** Sidon [6] introduced the following class of cosine trigonometric series: Let

$$(1.1) \quad \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

be a cosine series satisfying $a_k = o(1)$, $k \rightarrow \infty$. If there exists a sequence $\{A_k\}$ such that

$$(1.2) \quad A_k \downarrow 0, \quad k \rightarrow \infty,$$

$$(1.3) \quad \sum_{k=0}^{\infty} A_k < \infty,$$

$$(1.4) \quad |\Delta a_k| \leq A_k, \quad \forall k,$$

we say that (1.1) belongs to the class S .

Let the partial sums of (1.1) be denoted by $S_n(x)$ and $f(x) = \lim_{n \rightarrow \infty} S_n(x)$.

Recently, Garrett and Stanojević [3] proved that the partial Rees-Stanojević sums [5]

$$(1.5) \quad g_n(x) = \frac{1}{2} \sum_{k=0}^n \Delta a_k + \sum_{k=1}^n \sum_{j=k}^n \Delta a_j \cos kx$$

converge in the L^1 metric to (1.1) if and only if given $\varepsilon > 0$, there is a $\delta(\varepsilon) > 0$ such that

$$(1.6) \quad \int_0^{\delta} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right| dx < \varepsilon$$

for all $n \geq 0$. It has been shown in the same paper that the classical Young-Kolmogorov-Stanojević sufficient conditions for integrability of (1.1) imply (1.6).

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Generalising a classical result [1, p. 204], Teljakovskii [7] proved the following.

THEOREM A. *If (1.1) belongs to the class S , then a necessary and sufficient condition for L^1 convergence of (1.1) is $a_n \log n = o(1)$, $n \rightarrow \infty$.*

2. Lemmas. The proofs of our results are based upon the following lemmas.

LEMMA 1 (Fomin [2]). *If $|c_k| \leq 1$, then*

$$\int_0^\pi \left| \sum_{k=0}^n c_k \frac{\sin(k + 1/2)x}{2 \sin x/2} \right| dx \leq C(n + 1),$$

where C is a positive absolute constant.

LEMMA 2. *If (1.1) belongs to the class S , then*

$$g_n(x) = S_n(x) - a_{n+1}D_n(x),$$

where $D_n(x)$ denotes the Dirichlet kernel.

PROOF. Since (1.1) belongs to the class S , we have

$$a_k \rightarrow 0 \quad \text{and} \quad \sum_{k=0}^{\infty} |\Delta a_k| < \infty.$$

The conditions of Lemma 1 of Garrett and Stanojević [3] are thus satisfied and the result follows.

3. Main result. The main result of this paper reads:

THEOREM. *If (1.1) belongs to the class S , then (1.6) holds. Hence*

$$\|f - g_n\|_{L^1} = o(1), \quad n \rightarrow \infty.$$

PROOF. Making use of Abel's transformation and Lemma 1, we have

$$\begin{aligned} \int_0^\pi |f(x) - g_n(x)| dx &= \int_0^\pi \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right| dx \\ &= \int_0^\pi \left| \sum_{k=n+1}^{\infty} A_k \frac{\Delta a_k}{A_k} D_k(x) \right| dx \\ &= \int_0^\pi \left| \sum_{k=n+1}^{\infty} \Delta A_k \sum_{i=0}^k \frac{\Delta a_i}{A_i} D_i(x) \right| dx \\ &\leq \sum_{k=n+1}^{\infty} \Delta A_k \int_0^\pi \left| \sum_{i=0}^k \frac{\Delta a_i}{A_i} D_i(x) \right| dx \\ &\leq C \sum_{k=n+1}^{\infty} (k+1) \Delta A_k. \end{aligned}$$

(1.2) and (1.3) now imply the conclusion of the Theorem.

4. **Corollaries.** (i) Using Lemma 2, we notice that

$$\begin{aligned}\int_0^\pi |f(x) - S_n(x)| dx &= \int_0^\pi |f(x) - g_n(x) + g_n(x) - S_n(x)| dx \\ &\leq \int_0^\pi |f(x) - g_n(x)| dx + \int_0^\pi |g_n(x) - S_n(x)| dx \\ &\leq \int_0^\pi |f(x) - g_n(x)| dx + \int_0^\pi |a_{n+1} D_n(x)| dx\end{aligned}$$

and

$$\begin{aligned}\int_0^\pi |a_{n+1} D_n(x)| dx &= \int_0^\pi |g_n(x) - S_n(x)| dx \\ &\leq \int_0^\pi |f(x) - S_n(x)| dx + \int_0^\pi |f(x) - g_n(x)| dx.\end{aligned}$$

Since $\lim_{n \rightarrow \infty} \int_0^\pi |f(x) - g_n(x)| dx = 0$ by our Theorem and $\int_0^\pi |a_{n+1} D_n(x)| dx$ behaves like $a_{n+1} \log n$ for large values of n , Theorem A of Teljakovskii follows.

(ii) Let $a_k \rightarrow 0$ and $\sum_{k=1}^\infty (k+1) |\Delta^2 a_k| < \infty$. Then g_n converges to f in the metric space L since the trigonometric cosine series (1.1) with quasi-convex coefficients belongs to the class S if we choose $A_k = \sum_{m=k}^\infty |\Delta^2 a_m|$. This is Example 1 of [3].

5. **Remark.** In [4], Garrett and Stanojević proved (Corollary B, p. 70) that their Theorem B extends the Teljakovskii result.

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DEPARTMENT OF MATHEMATICS, ROHTAK UNIVERSITY, ROHTAK-124001, INDIA