SUBNORMAL GENERALIZED HAUSDORFF OPERATORS

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ABSTRACT. One type of generalized Hausdorff matrix is the lower triangular matrix with entries $h_{nk} = \binom{n+\alpha}{n-k} \Delta^{n-k} v_k$, where $\Delta v_n = v_n - v_{n+1}$, $v_n = \int_0^1 t^{n+\alpha} d\beta(t)$ for some $\beta(t)$ of bounded variation on $[0, 1]$ and for some $\alpha > 0$. The matrix H_α generated by $v_n = (n + \alpha + 1)^{-1}$ is shown to be a subnormal operator on l^2 if α is a nonnegative integer.

1. Introduction and summary. One type of generalized Hausdorff matrix is the lower triangular matrix with entries

$$h_{nk} = \binom{n+\alpha}{n-k} \Delta^{n-k} v_k,$$

where

$$\Delta v_n = v_n - v_{n+1}, \quad v_n = \int_0^1 t^{n+\alpha} d\beta(t)$$

for some $\beta(t)$ of bounded variation on $[0, 1]$ and for some $\alpha \geq 0$ (see [4]). Let H_α denote the generalized Hausdorff matrix generated by $v_n = \int_0^1 t^{n+\alpha} dt$. Then $h_{nk} = (n + \alpha + 1)^{-1}$, so that H_α is a factorable matrix. (A lower triangular matrix $A = (a_{nk})$ is called factorable if $a_{nk} = b_n c_k$ when $k \leq n$.) By [7, Theorem 2], any matrix which formally commutes with H_α must be triangular and, hence, by the proof of Theorem 198 in [3], must also be a generalized Hausdorff matrix. Thus, the operators on l^2 which commute with H_α are all generalized Hausdorff matrices. (For each α , the commutant of H_α is a closed algebra of operators. For $\alpha \neq \beta$, the only operators which commute with both H_α and H_β are the zero operator and the identity operator.)

Our main result is that, for every nonnegative integer α , H_α is a subnormal operator on l^2 and hence, by Yoshino's theorem [8] and the comments above, so is any generalized Hausdorff matrix which maps l^2 into l^2 and commutes with H_α . We explicitly describe a Borel probability measure μ_α on the closed unit disk (actually on the circles $|z - n/(n+1)| = 1/(n+1)$) such that $I - H_\alpha$ is unitarily equivalent to the operator $f(z) \rightarrow zf(z)$ acting on $H^2(\mu_\alpha)$, the closure of the polynomials in the metric of $L^2(\mu_\alpha)$. For $\alpha = 0$, this result was obtained in [5], [6]. For α a positive integer, the result is new. We are unable to determine whether H_α is subnormal for nonnegative noninteger values of α . Our approach relies on the methods and results of [1], [5], [6].

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2. Let $\alpha \geq 0$. We consider H_α as a linear transformation on l^2 or, equivalently, on the Hardy space H^2 of functions $f(z) = \sum a_n z^n$ analytic for $|z| < 1$ with $\sum |a_n|^2 < \infty$.

LEMMA 1. For each $\alpha \geq 0$, H_α is a bounded linear operator on l^2 with $\|H_\alpha\| = 2$. The point spectrum of H_α^* is the disk $\{\lambda: |1 - \lambda| < 1\}$. Each λ in this disk is a simple eigenvalue with corresponding eigenfunction ϕ_λ , where $\phi_\lambda(0) = 1$ and $\phi_\lambda(n) = \prod_{j=1}^n (1 - \lambda^{-1}(j + \alpha)^{-1})$, $n = 1, 2, 3, \dots$

PROOF. The first statement is proved in [4, p. 177]. The proof of the second statement is the same as the proof of Theorem 2 in [1].

If f belongs to l^2 , define F by $F(z) = \langle f, \phi_z \rangle_{l^2}$, for $|z| < 1$. The correspondence $f \leftrightarrow F$ is 1-1 since the elements ϕ_z , for $|z| < 1$, span a dense subset of l^2 . (For $\lambda = (\alpha + k)^{-1}$, $k = 1, 2, 3, \dots$, $\phi_\lambda(k) = 0$ for $n \geq k$ and $\phi_\lambda(n) \neq 0$ for $n \leq k - 1$.) Let $\mathcal{H}_\alpha$ denote all such functions F , and define $\|F\| = \|f\|_{l^2}$. Then $\mathcal{H}_\alpha$ is a Hilbert space of functions analytic for $|z| < 1$. (The analyticity follows since the reproducing kernel function for $\mathcal{H}_\alpha$, $k(w, z) = \langle \phi_w, \phi_z \rangle_{H^2}$ is the generalized hypergeometric function

$${}_3F_2(-\bar{w}/(1 - \bar{w}) + \alpha, -z/(1 - z) + \alpha, 1; \alpha + 1, \alpha + 1, 1),$$

which is analytic for $|z| < 1$.)

LEMMA 2. The operator $I - H_\alpha$ in l^2 is unitarily equivalent to the operator $f(z) \rightarrow zf(z)$ in $\mathcal{H}_\alpha$. The functions $1, (z - 1)^{-n}(z - \alpha/(1 + \alpha)) \dots (z - (n + \alpha - 1)/(n + \alpha))$, for $n \geq 1$, are an orthonormal basis for $\mathcal{H}_\alpha$.

PROOF. Lemma 2 follows from the proofs in [6, p. 216].

The mapping $z/(1 - z)$ takes the disk $|z| < 1$ onto the half-plane $\operatorname{Re} w > -\frac{1}{2}$. The inverse map is $w \rightarrow w/(1 + w)$. Let $\mathcal{H}_\alpha$ denote the set of functions G of the form $G(z) = F(z/(1 + z))$ for some F in $\mathcal{H}_\alpha$, $\operatorname{Re} z > -\frac{1}{2}$. Define a norm in $\mathcal{H}_\alpha$ by $\|G\| = \|F\|_{\mathcal{H}_\alpha}$. Then $\mathcal{H}_\alpha$ is a Hilbert space of functions analytic for $\operatorname{Re} z > -\frac{1}{2}$ with orthonormal basis $\psi_0 = 1$,

$$\psi_n(z) = (-1)^n \frac{\Gamma(\alpha + 1)}{\Gamma(n + \alpha + 1)} (z - \alpha)(z - \alpha - 1) \dots (z - \alpha - n + 1).$$

Since

$$\psi_n(z) = \frac{\Gamma(n + 1)\Gamma(\alpha + 1)}{\Gamma(n + \alpha + 1)} \phi_n(z - \alpha),$$

where

$$\phi_n(z) = (n!)^{-1} z(z - 1) \dots (z - n + 1),$$

it follows from [6, p. 218] that for $0 < a, b < 2$, a^z and b^z belong to $\mathcal{H}_\alpha$ and

$$\begin{aligned} \langle a^z, b^z \rangle_{\mathcal{H}_\alpha} &= (ab)^\alpha \sum_{n=0}^{\infty} \frac{\Gamma^2(n + \alpha + 1)}{\Gamma^2(n + 1)\Gamma^2(\alpha + 1)} \\ &= (ab)^{h-1} F(h, h; 1; (1 - a)(1 - b)), \end{aligned}$$

where $h = \alpha + 1$ and

$$F(h, h; 1; c) = \sum_{j=0}^{\infty} \frac{\Gamma^2(h+j)}{\Gamma^2(h)(j!)^2} c^j$$

is the hypergeometric function. This result, together with Theorem 1 below, will show that the norm in $\mathcal{H}_\alpha$ is given by integration with respect to a measure on the half-plane $\operatorname{Re} w \geq -\frac{1}{2}$. (We will retain the notation $h = \alpha + 1$ throughout.)

LEMMA 3. For all real numbers a, b and nonnegative integers x ,

$$\sum_{j=0}^m (-1)^j \frac{\Gamma(a+1)\Gamma(b+j)}{j!(m-j)!\Gamma(b)\Gamma(a-m+j+1)} = (-1)^m \frac{\Gamma(b-a+m)}{m!\Gamma(b-a)},$$

where we interpret $1/\Gamma(-n+1) = 0$ and $\Gamma(-c+n)/\Gamma(-c) = (-c)(-c+1)\dots(-c+n-1)$ for all positive integers n .

PROOF. The identity follows if one equates the coefficients of t^m in the binomial expansions of both sides of the identity $(1-t)^a(1-t)^{-b} = (1-t)^{-(b-a)}$.

COROLLARY. If $h > k$ are nonnegative integers, then

$$\sum_{j=0}^{h-k-1} (-1)^j \frac{\Gamma(h+k+j)}{j!(k+j)!\Gamma(h-j-k)} = (-1)^{h-k-1} \frac{\Gamma(h+k)}{k!\Gamma(h-k)}.$$

PROOF. Let $a = h-1$, $b = h+k$ and $n = h-k-1$ in Lemma 3. The result follows.

LEMMA 4. For h a positive integer and $|c| < 1$, we have the identity

$$F(h, h; 1; c) = (1-c)^{-h} \sum_{k=0}^{h-1} \frac{(-1)^{h-k-1} \Gamma(h+k)}{(k!)^2 \Gamma(h-k)} (1-c)^{-k}.$$

PROOF. Using formulas (4) on p. 57 and (22) on p. 64 of [2], and the identity $\Gamma(h)\Gamma(1-h)/\Gamma(1-h+j) = (-1)^j \Gamma(h-j)$, we get

$$\begin{aligned} F(h, h; 1; c) &= (1-c)^{-h} F(h, 1-h; 1; c/(c-1)) \\ &= (1-c)^{-h} \sum_{j=0}^{h-1} \frac{\Gamma(h+j)}{(j!)^2 \Gamma(h-j)} \left(\frac{c}{1-c} \right)^j. \end{aligned}$$

Expand $(c/(1-c))^j = (1/(1-c) - 1)^j$ by the binomial theorem and set $k = j - i$. The result then follows from the Corollary.

THEOREM 1. For each positive integer h and all $0 < a, b < 2$,

$$(ab)^{h-1} F(h, h, 1; (1-a)(1-b)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a^z b^{\bar{z}} d\beta_h(x, y),$$

where $\beta_h(x, y)$ is a joint probability measure defined by

$$d\beta_h(x, y) = \frac{|\Gamma(x+1+iy)|^4 dy d\mu(x)}{2\pi\Gamma^2(h)\Gamma(2x+2)|\Gamma(x-h+2+iy)|^2},$$

where $\mu(x) = 1$ if $x = (n-1)/2$ for some $n = 0, 1, 2, 3, \dots$, and $\mu(x) = 0$ otherwise.

PROOF. By Lemma 4,

$$\begin{aligned} & (ab)^{h-1}F(h, h, 1; (1-a)(1-b)) \\ &= (ab)^{h-1} \sum_{k=0}^{h-1} \frac{(-1)^{h-k-1}\Gamma(h+k)}{(k!)^2\Gamma(h-k)} (a+b-ab)^{-(h+k)}. \end{aligned}$$

By Theorem 1 in [5],

$$(a+b-ab)^{-(h+k)} = \sum \int_{-\infty}^{\infty} \frac{|\Gamma(x+h+k+iy)|^2 a^z b^{\bar{z}}}{2\pi\Gamma(2x+h+k+1)\Gamma(h+k)} dy,$$

where the sum is over all x of the form $x = (n-h-k)/2$, for $n = 0, 1, 2, \dots$. It follows, after a change of variables $x' = x + h - 1$, that

$$\begin{aligned} & (ab)^{h-1}F(h, h, 1; (1-a)(1-b)) \\ &= \sum_{k=0}^{h-1} \sum \int_{-\infty}^{\infty} \frac{(-1)^{h-k-1}|\Gamma(x+k+1+iy)|^2 a^z b^{\bar{z}}}{2\pi(k!)^2\Gamma(2x-h+k+3)\Gamma(h-k)} dy, \end{aligned}$$

where the inner sum is over all x of the form $x = \frac{1}{2}(n+h-k-2)$, for $n = 0, 1, 2, \dots$. Thus,

$$\begin{aligned} & (ab)^{h-1}F(h, h, 1; (1-a)(1-b)) \\ &= \frac{1}{2\pi} \sum \int_{-\infty}^{\infty} A_h(x, y) a^z b^{\bar{z}} dy \\ &+ \frac{1}{2\pi} \sum \int_{-\infty}^{\infty} B_h(x, y) a^z b^{\bar{z}} dy, \end{aligned}$$

where the first sum is over x in $\{-1/2, 0, 1/2, \dots, h/2 - 3/2\}$, the second sum is over x in $\{h/2 - 1/2, h/2, \dots\}$ and

$$\begin{aligned} A_h(x, y) &= \sum_{k=h-2-2x}^{h-1} \frac{(-1)^{h-k-1}|\Gamma(x+k+1+iy)|^2}{(k!)^2\Gamma(2x-h+k+3)\Gamma(h-k)}, \\ B_h(x, y) &= \sum_{k=0}^{h-1} \frac{(-1)^{h-k-1}|\Gamma(x+k+1+iy)|^2}{(k!)^2\Gamma(2x-h+k+3)\Gamma(h-k)}. \end{aligned}$$

Now, Saalschütz's Theorem [2, p. 188] states that for $m = 0, 1, 2, \dots$,

$$\begin{aligned} & \sum_{j=0}^m \frac{(-m)_j \Gamma(\alpha+j) \Gamma(\beta+j)}{j! (c)_j (1+\alpha+\beta-c-m)_j} \\ &= \frac{\Gamma(\alpha) \Gamma(\beta) \Gamma(c-\alpha+m) \Gamma(c-\beta+m)}{\Gamma(c-\alpha) \Gamma(c-\beta) (c)_m (c-\alpha-\beta)_m}, \end{aligned}$$

where $(c)_j = c(c+1) \dots (c+j-1)$ for $j \geq 1$, and $(c)_0 = 1$. Putting $m = h-1$, $\alpha = x+1+iy$, $\beta = x+1-iy$, $c = 2x-h+3$, and noting $\Gamma(h-k) = (-1)^k \Gamma(h)/(-h+1)_k$, it follows that

$$B_h(x, y) = |\Gamma(x+1+iy)|^4 \left\{ \Gamma^2(h) \Gamma(2x+2) |\Gamma(x-h+2+iy)|^2 \right\}^{-1}.$$

To simplify $A_h(x, y)$, put $k' = k-h+2+2x$, and then let $m = 2x+1$, $\alpha = h-1-x+iy$, $\beta = h-1-x-iy$, $c = h-2x-1$ in Saalschütz's Theorem to get

$$A_h(x, y) = |\Gamma(x+1+iy)|^2 |\Gamma(h-1-x+iy)|^2 \times \left\{ \Gamma^2(h) \Gamma(2x+2) |\Gamma(-x+iy)|^2 \right\}^{-1}.$$

It follows that $A_h(x, y) = B_h(x, y)$, which establishes the theorem except for the verification that β_h is a probability measure. But this follows from the observation that

$$\sum_{-\infty}^{\infty} \frac{1}{2\pi} B_h(x, y) dy = F(h, h, 1; 0) = 1 \quad \text{and} \quad B_h(x, y) > 0.$$

THEOREM 2. *If $\alpha = 0, 1, 2, 3, \dots$, then the operator $I - H_\alpha$ on l^2 is subnormal.*

PROOF. The proof is essentially the same as the proof of Theorem 2 in [5].

REMARK. The formula for $\beta_h(x, y)$ in Theorem 1 defines a positive measure whenever $h \geq 1$. However, the functions $\psi_n(z)$ which form an orthonormal basis for $\mathcal{K}_\alpha$ are neither orthogonal nor of norm 1 in the metric of $L^2(\alpha_h)$ unless h is a positive integer. Thus, the most natural candidate for a measure which determines the norm in $\mathcal{K}_\alpha$, for noninteger α , fails.

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REFERENCES

1. A. Brown, P. R. Halmos and A. L. Shields, *Cesaro operators*, Acta. Sci. Math. (Szeged) **26** (1965), 125-137.
2. A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher transcendental functions*, Vol. 1, McGraw-Hill, New York, 1953.
3. G. H. Hardy, *Divergent series*, Oxford Univ. Press, 1949.
4. A. Jakimovski, B. E. Rhoades and J. Tzimbalario, *Hausdorff matrices as bounded operators over l^p* , Math. Z. **138** (1974), 173-181.
5. E. Kay, H. Soul and D. Trutt, *Some subnormal operators and hypergeometric kernel functions*, J. Math. Anal. Appl. **53** (1976), 237-242.
6. T. L. Kriete and D. Trutt, *The Cesaro operator in l^2 is subnormal*, Amer. J. Math. **93** (1971), 215-225.
7. B. E. Rhoades, *Commutants of some Hausdorff matrices*, Pacific J. Math. **43** (1972), 715-719; **49** (1973), 617-619.
8. B. E. Rhoades and A. Wilansky, *Some commutants in $B(c)$ which are almost matrices*, Pacific J. Math. **49** (1973), 211-217.
9. T. Yoshino, *Subnormal operators with a cyclic vector*, Tohoku Math. J. **21** (1969), 47-55.

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