WEAKLY CLOSED m -ACCRETIVE OPERATORS¹

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ABSTRACT. Let A and B be weakly closed, nonlinear m -accretive (single-valued) operators in a reflexive Banach space X , and B_n be the Yosida approximation of B . Then the following condition is sufficient for the sum $A + B$ to be also m -accretive: For each $v \in X$, $\|B_n u_n\|$ is bounded as n tends to infinity, where u_n is defined by the equation $u_n + Au_n + B_n u_n = v$, $n = 1, 2, \dots$. Some related conditions are also provided.

1. Introduction. Let A and B be nonlinear (i.e. not necessarily linear) m -accretive operators in a Banach space X . Then the conditions under which the sum $A + B$ is also m -accretive have been studied by several authors (see e.g. [1], [2], [4], [6], [7]). In particular, Fitzgibbon [4] gave such a sufficient condition when X is reflexive and A, B are single-valued and weakly closed. His result is related to a theorem in Kato [6], i.e., it is established by showing that for each $x \in D(A + B)$ there exists a unique solution to the Cauchy problem: $u'(t) + (A + B)u(t) = 0$ for a.e. $t \geq 0$, $u(0) = x$.

In this note we shall prove a similar theorem, under a slightly generalized assumption, without using such techniques (see Theorem 2.4 and Corollary 2.5 below).

Throughout this note X denotes a reflexive Banach space, with the adjoint space X^* . The pairing between $u \in X$ and $f \in X^*$ is denoted by (u, f) . A single-valued operator A (with domain $D(A)$ and range $R(A)$) in X is said to be *accretive* if $\|(u + \lambda Au) - (v + \lambda Av)\| \geq \|u - v\|$ for all $u, v \in D(A)$ and $\lambda > 0$. Let F be the duality map from X to X^* : for each $w \in X$, $F(w) = \{f \in X^*; (w, f) = \|w\|^2 = \|f\|^2\}$. Then an operator A in X is accretive if and only if for all $u, v \in D(A)$ there is $f \in F(u - v)$ such that $\operatorname{Re}(Au - Av, f) \geq 0$; see [5]. An accretive operator is said to be *m -accretive* if $R(1 + \lambda A) = X$ for some (and hence for all) $\lambda > 0$.

2. The main result. Let A and B be m -accretive operators in X and B_n be the Yosida approximation of B : $B_n = n - n(1 + n^{-1}B)^{-1}$. Then B_n is everywhere defined and m -accretive. Since B_n is Lipschitz continuous, $A + B_n$ defined on $D(A)$ is also m -accretive. Consequently, for each $v \in X$ there exists $u_n \in D(A)$ such that

$$(1) \quad u_n + Au_n + B_n u_n = v, \quad n = 1, 2, \dots$$

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The following lemma is a nonlinear version of Lemma 2.1 in [10].

LEMMA 2.1. *Let A be a weakly closed, nonlinear m -accretive operator in a reflexive Banach space X , and A_n be its Yosida approximation.*

(i) *Let $\{u_n\}$ be a sequence in $D(A)$ such that $u_n \rightarrow u \in X$ ($\rightarrow$ denotes weak convergence). If the $\|Au_n\|$ are bounded, then $u \in D(A)$ and $Au_n \rightarrow Au$.*

(ii) *Let $\{w_n\}$ be a sequence in X such that $w_n \rightarrow u \in X$. If the $\|A_n w_n\|$ are bounded, then $u \in D(A)$ and $A_n w_n \rightarrow Au$.*

PROPOSITION 2.2. *Let X be reflexive, and A and B be weakly closed, nonlinear m -accretive operators in X . Let u_n be as in (1). If $\|B_n u_n\|$ is bounded as n tends to infinity, then $v \in R(1 + A + B)$.*

PROOF. Since u_n is defined by (1), we have

$$\begin{aligned} \|u_n\| &\leq \|u_n - u_1\| + \|u_1\| \leq \|(1 + A)u_n - (1 + A)u_1\| + \|u_1\| \\ &\leq \|B_n u_n\| + \|B_1 u_1\| + \|u_1\|, \quad n \geq 1. \end{aligned}$$

Therefore, $\|u_n\|$ is bounded by assumption. Since X is reflexive, there is a subsequence $\{u_m\}$ of $\{u_n\}$ such that $u_m \rightarrow u \in X$. Noting that both $\|Au_m\|$ and $\|B_m u_m\|$ are bounded in m by assumption, we see from Lemma 2.1 that $u \in D(A + B)$, $Au_m \rightarrow Au$ and $B_m u_m \rightarrow Bu$. Thus we obtain

$$v = (1 + A + B)u.$$

Q.E.D.

The next lemma, which generalizes a result in [4], is due to the referee.

LEMMA 2.3. *Let A and B be nonlinear accretive operators in a reflexive Banach space X . If A is m -accretive, then $A + B$ is accretive.*

PROOF. Let $u, v \in D(A)$. Noting that A is single-valued, we have that $\operatorname{Re}(Au - Av, f) \geq 0$ for all $f \in F(u - v)$ (see Miyadera [8, Theorem 3(i')]). Since B is accretive, this yields $\operatorname{Re}((A + B)u - (A + B)v, g) \geq 0$ for some $g \in F(u - v)$, whence $A + B$ is accretive. Q.E.D.

Our result is given by

THEOREM 2.4. *Let X be reflexive, and A and B be weakly closed, nonlinear m -accretive operators in X . Let u_n be as in (1). If for each $v \in X$, $\|B_n u_n\|$ is bounded as n tends to infinity, then $A + B$ is also m -accretive.*

PROOF. It suffices by Lemma 2.3 to show that $R(1 + A + B) = X$. But this follows immediately from assumption and Proposition 2.2. Q.E.D.

The following corollary was first proved by Fitzgibbon [4].

COROLLARY 2.5. *Let X be reflexive, and A and B be weakly closed, nonlinear m -accretive operators in X , with $D(A) \subset D(B)$. Assume that there exist constants $a \geq 0$, $0 \leq b < 1$ and $c \geq 0$ such that for all $u \in D(A)$,*

$$(2) \quad \|Bu\| \leq a\|u\| + b\|Au\| + c.$$

Then $A + B$ is also m -accretive.

PROOF. Let u_n be as in (1). Then it suffices to show that (2) implies the boundedness of $\|B_n u_n\|$ in n . Now it follows that

$$\begin{aligned}\|B_n u_n\| &\leq \|Bu_n\| \leq a\|u_n\| + b\|Au_n\| + c \\ &\leq a\|u_n\| + b\|(1 + A + B_n)u_n\| + b\|(1 + B_n)u_n\| + c.\end{aligned}$$

Therefore, we obtain

$$(1 - b)\|B_n u_n\| \leq b\|v\| + (a + b)\|u_n\| + c.$$

Thus, it remains to show that $\|u_n\|$ is bounded in n . Let $u \in D(A)$ be fixed. Then $\|u_n - u\| \leq \|v - (1 + A + B_n)u\|$. So we obtain

$$\|u_n\| \leq \|v\| + 2\|u\| + \|Au\| + \|Bu\|.$$

Q.E.D.

The next theorem is related to a theorem obtained by Barbu [2].

THEOREM 2.6. *Let X be reflexive, and A and B be m -accretive. Let A be linear, and B be nonlinear and weakly closed. Assume that $D(A)$ is invariant under $(1 + n^{-1}B)^{-1}$, $n \geq 1$, and there exists a constant $c \geq 0$ such that for all $u \in D(A)$,*

$$(3) \quad \|A(1 + n^{-1}B)^{-1}u\| \leq \|Au\| + cn^{-1}\|u\|, \quad n \geq 1.$$

Then the closure of $A + B$ is also m -accretive.

PROOF. $A + B$ is accretive by Lemma 2.3. Let $\overline{A + B}$ be the closure of $A + B$. Then $\overline{A + B}$ is also accretive. Since $R(1 + \overline{A + B})$ is closed, it suffices to show that $R(1 + A + B)$ is dense in X .

Let u_n be as in (1). Since (1) can be written

$$(A + n + 1)u_n = v + n(1 + n^{-1}B)^{-1}u_n,$$

we have, for $v \in D(A)$,

$$Au_n = (A + n + 1)^{-1}(Av + nA(1 + n^{-1}B)^{-1}u_n).$$

So we obtain, by (3),

$$\|Au_n\| \leq (n + 1)^{-1}\|Av\| + (n + 1)^{-1}n(\|Au_n\| + cn^{-1}\|u_n\|).$$

It follows that $\|Au_n\| \leq \|Av\| + c\|u_n\|$ and hence $\|B_n u_n\| \leq \|Av\| + \|v\| + (1 + c)\|u_n\|$. Now let $u \in D(A)$. Then $(1 + B)^{-1}u \in D(A + B)$ and, in the same way as in the proof of Corollary 2.5, we can prove that $\|u_n\|$ is bounded in n . Therefore, by Proposition 2.2 we see that $D(A) \subset R(1 + A + B)$. Since $D(A)$ is dense in X (see [11, p. 218]), so is $R(1 + A + B)$. Q.E.D.

3. Remark. The purpose of this section is to give a semilinear version of a theorem obtained in [3] and [9].

PROPOSITION 3.1. *Let X be reflexive, and A and B be m -accretive operators in X , with $D(A) \subset D(B)$. Let A be linear and B be nonlinear and weakly closed. Assume that there exist constants $a \geq 0$, $c \geq 0$ such that for all $u \in D(A)$,*

$$(4) \quad \|Bu\| \leq a\|u\| + \|Au\| + c.$$

If $\overline{A + B}$ is weakly closed, then $\overline{A + B}$ is m -accretive.

PROOF. As noted in the proof of Theorem 2.6, $\overline{A + B}$ is accretive. So it suffices to show that $R(1 + \overline{A + B}) = X$.

Since $\|Bu\| \leq a\|u\| + t^{-1}\|tAu\| + c$ for $1 < t \leq 2$, it follows from Corollary 2.5 that $tA + B$ is m -accretive. Consequently, for each $w \in X$ there exists a family $\{u(t)\}$ such that

$$(5) \quad (1 + tA + B)u(t) = w, \quad 1 < t \leq 2.$$

As in the proof of Corollary 2.5, $\|u(t)\|$ is bounded in t . Since X is reflexive, we can find a sequence $\{t_n\}$, $1 < t_n \leq 2$, and an element $u_0 \in X$ such that $t_n \rightarrow 1$ and

$$(6) \quad u(t_n) \rightharpoonup u_0 \quad \text{as } n \rightarrow \infty.$$

Next we set

$$(7) \quad v(t) = (1 + A + B)u(t).$$

It follows from (5) and (7) that

$$(8) \quad v(t) - w = (1 - t)Au(t), \quad 1 < t \leq 2.$$

By virtue of (4) we have

$$\begin{aligned} \|v(t) - w\| &= (t - 1)\|Au(t)\| \\ &\leq t\|Au(t)\| - (\|Bu(t)\| - a\|u(t)\| - c) \\ &\leq \|(tA + B)u(t)\| + a\|u(t)\| + c \\ &\leq \|w\| + (1 + a)\|u(t)\| + c. \end{aligned}$$

Consequently, $\{v(t_n)\}$ is a bounded sequence in X . We shall show that $v(t_n) \rightharpoonup w$ as $n \rightarrow \infty$. To this end, let A^* be the adjoint of A . Then we see from (8) that for $g \in D(A^*)$,

$$\begin{aligned} |(v(t_n) - w, g)| &= |(1 - t_n)(Au(t_n), g)| \\ &= (t_n - 1)|(u(t_n), A^*g)| \\ &\leq (t_n - 1)\|u(t_n)\| \|A^*g\| \rightarrow 0 \quad (n \rightarrow \infty). \end{aligned}$$

Since $D(A^*)$ is dense in X^* , it follows that $v(t_n) \rightharpoonup w$. In view of (6) and (7), we obtain from the weak closedness of $\overline{A + B}$, $u_0 \in D(\overline{A + B})$ and $(1 + \overline{A + B})u_0 = w$. Since $w \in X$ is arbitrary, $R(1 + \overline{A + B}) = X$. Q.E.D.

REFERENCES

1. H. Brezis, M. G. Crandall and A. Pazy, *Perturbations of nonlinear maximal monotone sets in Banach space*, Comm. Pure Appl. Math. **23** (1970), 123-144. MR **41** #2454.
2. V. Barbu, *Weak solutions for nonlinear functional equations in Banach spaces*, Ann. Mat. Pura Appl. **87** (1970), 87-110. MR **45** #7536.
3. P. R. Chernoff, *Perturbations of dissipative operators with relative bound one*, Proc. Amer. Math. Soc. **33** (1972), 72-74. MR **45** #5804.
4. W. E. Fitzgibbon, *Nonlinear perturbation of m -accretive operators*, Proc. Amer. Math. Soc. **44** (1974), 359-364. MR **49** #7856.
5. T. Kato, *Nonlinear semigroups and evolution equations*, J. Math. Soc. Japan **19** (1967), 508-520. MR **37** #1820.
6. ———, *Accretive operators and nonlinear evolution equations in Banach spaces*, Proc.

Sympos. Pure Math., vol. 18, part 1, Amer. Math. Soc., Providence, R. I., 1970, pp. 138–161. MR 42 #6663.

7. Y. Konishi, *A remark on perturbation of m -accretive operators in Banach space*, Proc. Japan Acad. 47 (1971), 452–455. MR 47 #7540.

8. I. Miyadera, *Some remarks on semi-groups of nonlinear operators*, Tôhoku Math. J. 23 (1971), 245–258. MR 45 #5805.

9. N. Okazawa, *A perturbation theorem for linear contraction semigroups on reflexive Banach spaces*, Proc. Japan Acad. 47 (1971), 947–949. MR 47 #2429.

10. ———, *Perturbations of linear m -accretive operators*, Proc. Amer. Math. Soc. 37 (1973), 169–174. MR 47 #2403.

11. K. Yosida, *Functional analysis*, 2nd ed., Grundlehren math. Wiss., Band 123, Springer-Verlag, Berlin and New York, 1968. MR 39 #741.

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